

Structure theorems for singular minimal laminations

William H. Meeks III*

Joaquín Pérez

Antonio Ros,[†]

November 24, 2016

Abstract

We apply the local removable singularity theorem for minimal laminations [31] and the local picture theorem on the scale of topology [23] to obtain two descriptive results for certain possibly *singular minimal laminations* of \mathbb{R}^3 . These two global structure theorems will be applied in [21] to obtain bounds on the index and the number of ends of complete, embedded minimal surfaces of fixed genus and finite topology in \mathbb{R}^3 , and in [22] to prove that a complete, embedded minimal surface in \mathbb{R}^3 with finite genus and a countable number of ends is proper.

Mathematics Subject Classification: Primary 53A10, Secondary 49Q05, 53C42

Key words and phrases: Minimal surface, stability, curvature estimates, local picture, minimal lamination, removable singularity, minimal parking garage structure, injectivity radius, locally simply connected.

1 Introduction.

The analysis of singularities of embedded minimal surfaces and more generally of minimal laminations in three-manifolds is a transcendental open problem in minimal surface theory. Theory developed by Colding and Minicozzi [4, 5, 6, 7, 8, 9] and subsequent applications by Meeks and Rosenberg [32, 33] and Meeks, Pérez and Ros [24, 25, 28] demonstrate the importance of the analysis of singularities of minimal laminations. Removable singularity theorems in [31] have been instrumental in obtaining classification results [29] for CMC foliations of \mathbb{R}^3 and \mathbb{S}^3 with a countable set of singularities, in studying dynamical properties of the space of properly embedded minimal surfaces in \mathbb{R}^3 [30], and in deriving local pictures on the extrinsic geometry of an embedded minimal surface around points of arbitrarily small injective radius [23].

In this paper we will improve the understanding of singularities of minimal laminations in \mathbb{R}^3 with two new results on the global structure of these objects. In the first result, Theorem 1.4, we describe the possible limits (after extracting a subsequence) of a sequence of embedded minimal surfaces with *locally positive injectivity radius*¹ in the complement of a countable closed set of \mathbb{R}^3 . The second result, Theorem 4.1, describes the structure of a singular minimal lamination of

*This material is based upon work for the NSF under Award No. DMS - 1309236. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

[†]The second and third authors were supported in part by the MINECO/FEDER grant no. MTM2014-52368-P.

¹See Definition 1.3 for this notion.

\mathbb{R}^3 whose singular set is countable. Both results depend on the local theory of embedded minimal surfaces and minimal laminations developed in [23, 30, 31], and on the previously mentioned work of Colding and Minicozzi. For the definition and the general theory of minimal laminations, see for instance [20, 26, 27, 31, 32, 33].

We next give a formal definition of a singular lamination and of the set of singularities associated to a leaf of a singular lamination. Given an open set $A \subset \mathbb{R}^3$ and a subset $B \subset A$, we will denote by \overline{B}^A the closure of B in A . In the case $A = \mathbb{R}^3$, we simply denote $\overline{B}^{\mathbb{R}^3}$ by \overline{B} .

Definition 1.1 A *singular lamination* of an open set $A \subset \mathbb{R}^3$ with *singular set* $\mathcal{S} \subset A$ is the closure $\overline{\mathcal{L}}^A$ of a lamination \mathcal{L} of $A - \mathcal{S}$, such that for each point $p \in \mathcal{S}$, then $p \in \overline{\mathcal{L}}^A$, and in every open neighborhood $U_p \subset A$ of p , $\overline{\mathcal{L}}^A \cap U_p$ fails to have an induced lamination structure in U_p . It then follows that \mathcal{S} is closed in A . The singular lamination $\overline{\mathcal{L}}^A$ is said to be *minimal* if the leaves of the related lamination \mathcal{L} of $A - \mathcal{S}$ are minimal surfaces.

For a leaf L of \mathcal{L} , we call a point $p \in \overline{L}^A \cap \mathcal{S}$ a *singular leaf point of L* if there exists an open set $V \subset A$ containing p such that $L \cap V$ is closed in $V - \mathcal{S}$. We let \mathcal{S}_L denote the *set of singular leaf points of L* . Finally, we define

$$\overline{\mathcal{L}}^A(L) = L \cup \mathcal{S}_L \quad (1)$$

to be the *leaf of $\overline{\mathcal{L}}^A$ associated to the leaf L of \mathcal{L}* . In the case $A = \mathbb{R}^3$, we simply denote $\overline{\mathcal{L}}^A(L)$ by $\overline{\mathcal{L}}(L)$. In particular, the leaves of $\overline{\mathcal{L}}^A$ are of one of the following two types.

- If for a given leaf L in \mathcal{L} we have $\overline{L}^A \cap \mathcal{S} = \emptyset$, then L a leaf of $\overline{\mathcal{L}}^A$.
- If for a given leaf L in \mathcal{L} we have $\overline{L}^A \cap \mathcal{S} \neq \emptyset$, then $\overline{\mathcal{L}}^A(L)$ is a leaf of $\overline{\mathcal{L}}^A$.

Note that since \mathcal{L} is a lamination of $A - \mathcal{S}$, then $\overline{\mathcal{L}}^A = \mathcal{L} \cup \mathcal{S}$. Hence, the closure $\overline{\mathcal{L}}$ of \mathcal{L} when considered to be a subset of \mathbb{R}^3 is the set $\overline{\mathcal{L}} = \mathcal{L} \cup \mathcal{S} \cup (\partial A \cap \overline{\mathcal{L}})$.

In contrast to the behavior of (regular) laminations, it is possible for distinct leaves of a singular lamination to intersect. In Section 2 we will give an example that illustrates this phenomenon.

Definition 1.2 With the notation in Definition 1.1, a leaf $\overline{\mathcal{L}}^A(L) = L \cup \mathcal{S}_L$ of $\overline{\mathcal{L}}^A$ is said to be a *limit leaf* of $\overline{\mathcal{L}}^A$ if the related leaf $L \in \mathcal{L}$ is a limit leaf of \mathcal{L} (i.e., there exists a point $p \in L$ that is a limit in A of a sequence of points $p_n \in L_n$, where L_n is a leaf of \mathcal{L} for all n , and if $L_n = L$ after passing to a subsequence, then the sequence p_n does not converge to p in the intrinsic topology of L). We will denote by $\text{Lim}(\overline{\mathcal{L}}^A)$ the set of limit leaves of $\overline{\mathcal{L}}^A$.

Throughout the paper, $\mathbb{B}(p, R)$ will denote the open Euclidean ball of radius $R > 0$ centered at a point $p \in \mathbb{R}^3$, $\mathbb{B}(R) = \mathbb{B}(\vec{0}, R)$, $\mathbb{S}^2(p, R) = \partial \mathbb{B}(p, R)$ and $\mathbb{S}^2(R) = \mathbb{S}^2(\vec{0}, R)$. For a surface $\Sigma \subset \mathbb{R}^3$, K_Σ will denote its Gaussian curvature function.

Definition 1.3 Let $\{M_n\}_n$ be a sequence of surfaces (possibly with boundary) in an open set $A \subset \mathbb{R}^3$. We will say that $\{M_n\}_n$ has *locally positive injectivity radius in A* , if for every $q \in A$, there exists $\varepsilon_q > 0$ and $n_q \in \mathbb{N}$ such that for $n > n_q$, the restricted functions $(I_{M_n})|_{\mathbb{B}(q, \varepsilon_q) \cap M_n}$ are uniformly bounded away from zero, where I_{M_n} is the injectivity radius function of M_n .

Note that if the surfaces M_n have boundary and $\{M_n\}_n$ has locally positive injectivity radius in A , then for any $p \in A$ there exists $\varepsilon_p > 0$ and $n_p \in \mathbb{N}$ such that $\partial M_n \cap \mathbb{B}(p, \varepsilon_p) = \emptyset$ for $n > n_p$, i.e., points in the boundary of M_n must eventually diverge in space or accumulate to points in the complement of A .

By Proposition 1.1 in Colding and Minicozzi [8], the property that a sequence of embedded minimal surfaces $\{M_n\}_n$ has locally positive injectivity radius in an open set A is equivalent to the property that $\{M_n\}_n$ is *locally simply connected in A* , in the sense that around any point $q \in A$, we can find $\delta_q > 0$ such that $\mathbb{B}(q, \delta_q) \subset A$ and for n sufficiently large, $\mathbb{B}(q, \delta_q)$ intersects M_n in components that are disks with boundaries in $\mathbb{S}^2(q, \delta_q)$.

Theorem 1.4 *Suppose W is a countable closed subset of \mathbb{R}^3 and $\{M_n\}_n$ is a sequence of embedded minimal surfaces (possibly with boundary) in $A = \mathbb{R}^3 - W$, that has locally positive injectivity radius in A . Then, there exist a closed subset $\mathcal{S}^A \subset A$, a minimal lamination \mathcal{L} of $A - \mathcal{S}^A$ and a subset $S(\mathcal{L}) \subset \mathcal{L}$ (in particular, $S(\mathcal{L}) \cap \mathcal{S}^A = \emptyset$) such that after replacing by a subsequence, $\{M_n\}_n$ converges C^α , for all $\alpha \in (0, 1)$, on compact subsets of $A - (S(\mathcal{L}) \cup \mathcal{S}^A)$ to \mathcal{L} (here $S(\mathcal{L})$ is the singular set of convergence² of $\{M_n\}_n$ to \mathcal{L}), and the closure of \mathcal{L} in A has the structure of a possibly singular minimal lamination of A with singular set \mathcal{S}^A :*

$$\overline{\mathcal{L}}^A = \mathcal{L} \cup \mathcal{S}^A.$$

Furthermore, the closure $\overline{\mathcal{L}}$ in \mathbb{R}^3 of \mathcal{L} has the structure of a possibly singular minimal lamination of \mathbb{R}^3 , with the singular set \mathcal{S} of $\overline{\mathcal{L}}$ satisfying $\mathcal{S}^A \subset \mathcal{S} \subset \mathcal{S}^A \cup (W \cap \overline{\mathcal{L}})$, and:

1. The set \mathcal{P} of planar leaves in $\overline{\mathcal{L}}$ forms a closed subset of \mathbb{R}^3 .
2. The set $\text{Lim}(\overline{\mathcal{L}})$ of limit leaves of $\overline{\mathcal{L}}$ forms a closed set in \mathbb{R}^3 and satisfies $\text{Lim}(\overline{\mathcal{L}}) \subset \mathcal{P}$.
Furthermore, if $L = \overline{\mathcal{L}}(L_1) = L_1 \cup S_{L_1}$ is a leaf of $\overline{\mathcal{L}}$ (here L_1 is the related leaf of the regular lamination associated to $\overline{\mathcal{L}}$, see (1)) and $A \cap S_{L_1} \neq \emptyset$, then L is a limit leaf of $\overline{\mathcal{L}}$. In particular, every singular leaf point of a non-flat leaf of $\overline{\mathcal{L}}$ belongs to W .
3. If P is a plane in $\mathcal{P} - \text{Lim}(\overline{\mathcal{L}})$, then there exists $\delta > 0$ such that $P(\delta) \cap \overline{\mathcal{L}} = P$, where $P(\delta)$ is the δ -neighborhood of P . In particular, $\mathcal{S} \cap [\mathcal{P} - \text{Lim}(\overline{\mathcal{L}})] = \emptyset$.
4. For each point $q \in \mathcal{S}^A \cup S(\mathcal{L})$, there passes a plane $P_q \in \text{Lim}(\overline{\mathcal{L}})$. Furthermore, P_q intersects $\mathcal{S}^A \cup S(\mathcal{L}) \cup W$ in a closed countable set.
5. Through each point of $p \in W \cap \overline{\mathcal{L}}$ satisfying one of the conditions 5.1, 5.2 below, there passes a planar leaf P_p in \mathcal{P} .
 - 5.1. For all $k \in \mathbb{N}$, there exists $\varepsilon_k \in (0, \frac{1}{k})$ and an open subset Ω_k of $\mathbb{B}(p, \varepsilon_k)$ such that $W \cap \mathbb{B}(p, \varepsilon_k) \subset \Omega_k \subset \overline{\Omega}_k \subset \mathbb{B}(p, \varepsilon_k)$ and the area of $M_n \cap [\mathbb{B}(p, \varepsilon_k) - \overline{\Omega}_k]$ diverges to infinity as $n \rightarrow \infty$ (in this case, the convergence of the M_n to P_p has infinite multiplicity).
 - 5.2. The convergence of the M_n to some leaf of \mathcal{L} having p in its closure is of finite multiplicity greater than one.

² $S(\mathcal{L})$ is the set of points $x \in \mathcal{L}$ such that $\sup_{n \in \mathbb{N}} |K_{M_n \cap \mathbb{B}(x, \varepsilon)}|$ is not bounded for any $\varepsilon > 0$.

6. Suppose that there exists a leaf $L = L_1 \cup S_{L_1}$ of $\overline{\mathcal{L}}$ that is not contained in \mathcal{P} , where L_1 is the related leaf of the regular lamination $\mathcal{L}_1 := \overline{\mathcal{L}} - S$ of $\mathbb{R}^3 - S$ and S_{L_1} is the set of singular leaf points of L_1 . Then, $L \cap (S^A \cup S(\mathcal{L})) = \emptyset$ (note that L might contain singular points which necessarily belong to W), the convergence of portions of the M_n to L_1 is of multiplicity one, and one of the following two possibilities holds:

- 6.1. L is proper³ in \mathbb{R}^3 , $S = S_{L_1} \subset W$ and L is the unique leaf of $\overline{\mathcal{L}}$.
- 6.2. L is not proper in \mathbb{R}^3 and $\mathcal{P} \neq \emptyset$. In this case, $\overline{\mathcal{L}}$ has the structure of a possibly singular minimal lamination of \mathbb{R}^3 with a countable set of singularities, there exists a subcollection $\mathcal{P}(L) \subset \mathcal{P}$ consisting of one or two planes such that $\overline{\mathcal{L}} = L \cup \mathcal{P}(L)$, L is proper in a component $C(L)$ of $\mathbb{R}^3 - \mathcal{P}(L)$ and $C(L) \cap \overline{\mathcal{L}} = L$. Furthermore:
 - a. Every open ε -neighborhood $P(\varepsilon)$ of a plane $P \in \mathcal{P}(L)$ intersects L_1 in a connected surface with unbounded Gaussian curvature.
 - b. If some ε -neighborhood $P(\varepsilon)$ of a plane $P \in \mathcal{P}(L)$ intersects L_1 in a surface with finite genus, then $P(\varepsilon)$ is disjoint from the singular set of $\overline{\mathcal{L}}$.
 - c. L_1 has infinite genus.

In particular, $\overline{\mathcal{L}}$ is the disjoint union of its leaves, regardless of which case 6.1 or 6.2 occurs (if case 6.2 occurs, then each leaf of $\overline{\mathcal{L}}$ is either a plane or a minimal surface possibly with singularities in W , that is proper² in an open halfspace or slab of \mathbb{R}^3).

7. Suppose that the surfaces M_n have uniformly bounded genus and $S \cup S(\mathcal{L}) \neq \emptyset$. Then:

- 7.1. $\overline{\mathcal{L}} = \mathcal{P}$ and so, $S = \emptyset$.
- 7.2. $\overline{\mathcal{L}}$ contains a foliation \mathcal{F} of an open slab of \mathbb{R}^3 by planes and $S(\mathcal{L}) \cap \mathcal{F}$ consists of one or two straight line segments orthogonal to the planes in \mathcal{F} , where each line segment intersects every plane in \mathcal{F} . Furthermore, if there are 2 different line segments in $S(\mathcal{L}) \cap \mathcal{F}$, then in the related limit minimal parking garage structure of the slab, the two multivalued graphs occurring inside the surfaces M_n along $S(\mathcal{L}) \cap \mathcal{F}$ are oppositely handed.
- 7.3. If the M_n are compact with boundary, then $\overline{\mathcal{L}}$ is a foliation of \mathbb{R}^3 by planes and $\overline{S(\mathcal{L})}$ consists of one or two complete lines orthogonal to the planes in this foliation.

In item 7.2 of Theorem 1.4 we mentioned the “related limit minimal parking garage structure of the slab”; we refer the reader to our paper [23] for the notion of limit minimal parking garage structure of \mathbb{R}^3 (see Colding and Minicozzi [9] for a related discussion). Limit minimal parking garage structures in [23] are foliations of \mathbb{R}^3 by planes, that appear as the limit outside a discrete set of lines orthogonal to the planes, of certain sequences of embedded minimal surfaces that are uniformly locally simply connected in \mathbb{R}^3 . The fact that the sequence $\{M_n\}_n$ in Theorem 1.4 is only locally simply connected outside W is what might produce a foliated slab rather than the whole \mathbb{R}^3 . In spite of this problem that arises from W , we feel that our language here appropriately describes the behavior of the limit configuration, since if \mathcal{F} is a union of planar leaves of $\overline{\mathcal{L}}$ that

³As leaves of $\overline{\mathcal{L}}$ may have singularities, properness of such a leaf $L = L_1 \cup S_{L_1}$ just means that L is a closed set of \mathbb{R}^3 , or equivalently, S_{L_1} is closed in \mathbb{R}^3 and L_1 is a proper surface in the complement of S_{L_1} .

forms an open slab and $\mathcal{F} \cap S(\mathcal{L}) \neq \emptyset$, then $\mathcal{F} \cap \mathcal{S} = \emptyset$ and for n large, $M_n \cap \mathcal{F}$ has the appearance of a parking garage surface on large compact domains of this open slab, away from W . In Example 2.3 below we will exhibit a parking garage structure of the upper halfspace of \mathbb{R}^3 .

Regarding applications, Theorem 1.4 will be crucial in the proof of the following results:

- (I) In Theorem 4.1 below we will describe the structure of any singular minimal lamination $\bar{\mathcal{L}} = \mathcal{L} \cup \mathcal{S}$ of \mathbb{R}^3 with countable singular set \mathcal{S} . Roughly speaking, either $\bar{\mathcal{L}}$ consists of a single leaf which is a properly embedded minimal surface ($\mathcal{S} = \emptyset$ in this case), or $\bar{\mathcal{L}}$ consists of a closed family \mathcal{P} of parallel planes that contains all limit leaves of $\bar{\mathcal{L}}$, together with non-flat leaves in $\mathbb{R}^3 - \mathcal{P}$, each of which has infinite genus, unbounded Gaussian curvature and is properly embedded in an open slab or halfspace bounded by one of two planes in \mathcal{P} , and limits to these planes (in particular, non-flat leaves are not proper in \mathbb{R}^3).
- (II) In [21] we will apply Theorem 1.4 to prove that for each $g \in \mathbb{N} \cup \{0\}$, there exists a bound on the number of ends of a complete, embedded minimal surface in \mathbb{R}^3 with finite topology and genus at most g . This topological boundedness result implies that the stability index of a complete, embedded minimal surface of finite index in \mathbb{R}^3 has an upper bound that depends only on its finite genus.
- (III) We will use Theorem 1.4 in [22] to show that a connected, complete, embedded minimal surface in \mathbb{R}^3 with an infinite number of ends, finite genus and compact (possibly empty) boundary, is proper if and only if it has a countable number of limit ends, if and only if it has one or two limit ends, and when the boundary of the proper surface is empty, then it has exactly two limit ends (limit ends are the limit points in the space of ends endowed with its natural topology). Both (II) and (III) were announced a long time ago, but we found some problems in the original proof that have finally been resolved by applications of results in the present paper.

Besides the above applications, Theorems 1.4 and 4.1 provide geometrical insight for possibly resolving the following fundamental conjecture, at least when the set \mathcal{S} is countable.

Conjecture 1.5 (Fundamental Singularity Conjecture)

Suppose $\mathcal{S} \subset \mathbb{R}^3$ is a closed set whose one-dimensional Hausdorff measure is zero. If \mathcal{L} is a minimal lamination of $\mathbb{R}^3 - \mathcal{S}$, then the closure $\bar{\mathcal{L}}$ has the structure of a minimal lamination of \mathbb{R}^3 .

Since the union of a catenoid with a plane passing through its waist circle is a singular minimal lamination of \mathbb{R}^3 whose singular set is the intersecting circle, the above conjecture represents the strongest possible removable singularity conjecture.

The paper is organized as follows. In Section 2 we give examples of singular minimal laminations and obtain some results to be used in the proof of the main Theorem 1.4; these auxiliary results are based on the local removable singularity theorem [31] and the stable limit leaf theorem for the limit leaves of a minimal lamination [26, 27]. In Section 3 we prove Theorem 1.4. Section 4 contains the statement and proof of the application (I) (Theorem 4.1) listed above. In the final Section 5 we will describe the subsequential limit of a sequence $\{M_n\}_n$ of compact embedded minimal surfaces of genus at most $g \in \mathbb{N} \cup \{0\}$, with boundaries ∂M_n diverging in \mathbb{R}^3 ,

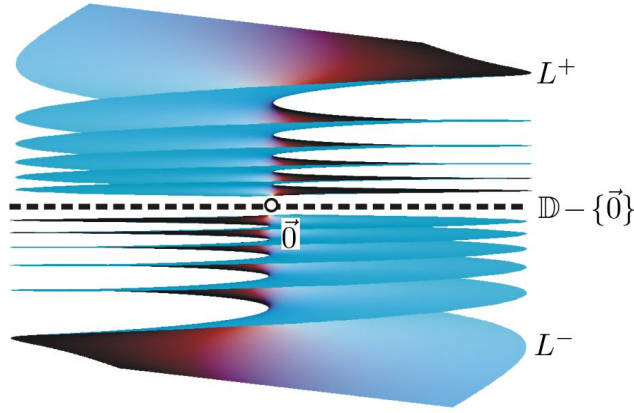


Figure 1: The origin is a singular leaf point of the horizontal disk passing through it, but not of the two nonproper spiraling leaves L^+, L^- .

provided that the M_n contain disks that converge C^2 to a nonflat minimal disk: a subsequence converges smoothly on compact subsets of \mathbb{R}^3 with multiplicity one to a connected nonflat minimal surface of genus at most g which is properly embedded in \mathbb{R}^3 , has bounded Gaussian curvature, and either it has finite total curvature, or is a helicoid with handles or a two-limit-ended surface.

2 Preliminaries.

We start by giving an example of a singular minimal lamination whose leaves intersect. We will use the notation introduced in Definition 1.1.

Example 2.1 The union of two transversal planes $\Pi_1, \Pi_2 \subset \mathbb{R}^3$ is a singular lamination $\overline{\mathcal{L}}$ of $A = \mathbb{R}^3$ with singular set \mathcal{S} being the line $\Pi_1 \cap \Pi_2$. In this example, Definition 1.1 yields a related lamination \mathcal{L} of $\mathbb{R}^3 - \mathcal{S}$ with four leaves that are open halfplanes in $\Pi_i - (\Pi_1 \cap \Pi_2)$, $i = 1, 2$, $\overline{\mathcal{L}}$ has four leaves that are the associated closed halfplanes that intersect along \mathcal{S} ; thus, $\overline{\mathcal{L}}$ is *not* the disjoint union of its leaves: every point in \mathcal{S} is a singular leaf point of each of the four leaves of \mathcal{L} .

In our second example, the leaves of the singular minimal lamination will not intersect.

Example 2.2 Colding and Minicozzi [3] constructed a singular minimal lamination $\overline{\mathcal{L}}_1$ of the open unit ball $A = \mathbb{B}(1) \subset \mathbb{R}^3$ with singular set \mathcal{S}_1 being the origin $\{\vec{0}\}$; the related (regular) lamination \mathcal{L}_1 of $\mathbb{B}(1) - \{\vec{0}\}$ consists of three leaves, which are the punctured unit disk $\mathbb{D} - \{\vec{0}\} = \{(x_1, x_2, 0) \mid 0 < x_1^2 + x_2^2 < 1\}$ and two nonproper disks $L^+ \subset \{x_3 > 0\}$ and $L^- \subset \{x_3 < 0\}$ that spiral to $\mathbb{D} - \{\vec{0}\}$ from opposite sides, see Figure 1. In this case, $\vec{0}$ is a singular leaf point of $\mathbb{D} - \{\vec{0}\}$ (hence $\overline{\mathcal{L}}_1^A(\mathbb{D} - \{\vec{0}\})$ equals the unit disk \mathbb{D}), but $\vec{0}$ is not a singular leaf point of either L^+ or L^- because $L^+ \cap V$ fails to be closed in $V - \mathcal{S}_1$ for any open set $V \subset \mathbb{B}(1)$ containing $\vec{0}$. Thus, $\overline{\mathcal{L}}_1^A(L^+) = L^+$ and analogously $\overline{\mathcal{L}}_1^A(L^-) = L^-$. Hence, $\overline{\mathcal{L}}_1$ is the disjoint union of its leaves in this case.

Example 2.2 is produced as a limit of a sequence of embedded minimal surfaces. We will use this sequence to produce an example of a parking garage structure of the upper halfspace of \mathbb{R}^3 , as announced right after the statement of Theorem 1.4.

Example 2.3 Colding and Minicozzi [3] proved the existence of a sequence $\{D_n\}_n$ of compact minimal disks contained in the closed unit ball $\mathbb{B}(1)$ of \mathbb{R}^3 , with $\partial D_n \subset \mathbb{S}^2(1)$, such that $\{\text{Int}(D_n)\}_n$ converges as $n \rightarrow \infty$ to the singular minimal lamination $\overline{\mathcal{L}}_1 = \mathcal{L}_1 \cup \mathcal{S}_1$ of $\mathbb{B}(1)$ that appears in Example 2.2. By the local removable singularity theorem for minimal laminations (see Theorem 1.1 in [31] or see Theorem 2.4 below), the Gaussian curvature function $K_{\mathcal{L}_1}$ of \mathcal{L}_1 satisfies that $|K_{\mathcal{L}_1}|R^2$ is unbounded in arbitrarily small neighborhoods of $\vec{0}$, where $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Defining $\lambda_n = |p_n|^{-1/2}$, where p_n are points in D_n such that $p_n \rightarrow \vec{0}$ and $|K_{D_n}|(p_n)|p_n|^2 \rightarrow \infty$, then the Gaussian curvature of the homothetically expanded disks $\lambda_n D_n$ blows up around $\vec{0}$. By Theorem 0.1 in Colding and Minicozzi [7], after passing to a subsequence, the $\lambda_n D_n$ converge to a foliation \mathcal{F} of \mathbb{R}^3 by planes (which can be proved to be horizontal due to the properties of D_n in [3]), with singular set of convergence $S(\mathcal{F})$ being a transverse Lipschitz curve to the planes in the foliation, which in fact is the x_3 -axis by the $C^{1,1}$ -regularity theorem of Meeks [18]). It then follows that $M_n = \lambda_n L^+ \subset \{x_3 > 0\}$ is a nonproper, embedded minimal disk, and the sequence $\{M_n\}_n$ has locally positive injectivity radius in $\mathbb{R}^3 - \{\vec{0}\}$ and converges in \mathbb{R}^3 minus the nonnegative x_3 -axis to the minimal lamination \mathcal{L} of $\mathbb{R}^3 - \{\vec{0}\}$ by all horizontal planes with positive heights together with the (x_1, x_2) -plane punctured at $\vec{0}$, whose singular set of convergence $S(\mathcal{L})$ is the positive x_3 -axis. In this case, $W = \{\vec{0}\}$ and $\overline{\mathcal{L}}$ is the foliation of the closed upper halfspace of \mathbb{R}^3 by horizontal planes; in particular $\mathcal{S} = \emptyset$.

Conjecture 1.5 stated in the Introduction has a global nature, because there exist interesting minimal laminations of the open unit ball in \mathbb{R}^3 punctured at the origin that do not extend across the origin, see Figure 1 and also see Examples I and II in Section 2 in [31]. In Example III of Section 2 of [31] we described a rotationally invariant global minimal lamination of hyperbolic three-space \mathbb{H}^3 , which has a similar unique isolated singularity. The existence of this global singular minimal lamination of \mathbb{H}^3 demonstrates that the validity of Conjecture 1.5 must depend on the metric properties of \mathbb{R}^3 . However, in [29] and [31], we obtained a remarkable local removable singularity result, valid in any Riemannian three-manifold N for certain possibly singular laminations all whose leaves have the same constant mean curvature. Since we will apply this theorem and a related corollary repeatedly in the minimal case, we give their complete statements below in this minimal case.

Given a three-manifold N and a point $p \in N$, we will denote by $B_N(p, r)$ the metric ball of center p and radius $r > 0$.

Theorem 2.4 (Local Removable Singularity Theorem [31]) *A minimal lamination \mathcal{L} of a punctured ball $B_N(p, r) - \{p\}$ in a Riemannian three-manifold N extends to a minimal lamination of $B_N(p, r)$ if and only if there exists a positive constant c such that $|\sigma_{\mathcal{L}}|d < c$ in some subball centered at p , where $|\sigma_{\mathcal{L}}|$ is the norm of the second fundamental form of the leaves of \mathcal{L} and d is the distance function in N to p .*

The following result is a consequence of Theorem 2.4; see Corollary 7.1 in [31] for a proof.

Corollary 2.5 *Suppose that N is a (not necessarily complete) Riemannian three-manifold. If $W \subset N$ is a closed countable subset and \mathcal{L} is a minimal lamination of $N - W$, then the closure of any collection of its stable leaves extends across W to a minimal lamination of N consisting of stable minimal surfaces. In particular,*

1. *The closure $\overline{\text{Stab}(\mathcal{L})}$ in N of the collection of stable leaves of \mathcal{L} is a minimal lamination of N whose leaves are stable minimal surfaces.*
2. *The closure $\overline{\text{Lim}(\mathcal{L})}$ in N of the sublamination $\text{Lim}(\mathcal{L})$ of limit leaves of \mathcal{L} is a sublamination of $\overline{\text{Stab}(\mathcal{L})}$.*
3. *If \mathcal{L} is a minimal foliation of $N - W$, then \mathcal{L} extends across W to a minimal foliation of N .*

Theorems 1.4 and 4.1 deal with the structure of certain possibly singular minimal laminations of \mathbb{R}^3 . In both results, the singular laminations can be expressed as a disjoint union of its possibly singular minimal leaves (see the last statement of item 6 of Theorem 1.4 and of item 5 of Theorem 4.1). The key result for proving this disjointness property of leaves is the next proposition; it gives a condition under which two different leaves of a singular minimal lamination cannot share a singular leaf point.

Proposition 2.6 *Let $\overline{\mathcal{L}}^A$ be a singular minimal lamination of an open set $A \subset \mathbb{R}^3$, with countable singular set \mathcal{S} and related (regular) lamination \mathcal{L} of $A - \mathcal{S}$. Then, any singular point is a singular leaf point of at most one leaf of $\overline{\mathcal{L}}^A$.*

Proof. Reasoning by contradiction, suppose that $p \in \mathcal{S}$ is a singular leaf point of two different leaves $\overline{\mathcal{L}}^A(L_1), \overline{\mathcal{L}}^A(L_2)$ of $\overline{\mathcal{L}}^A$, associated to leaves L_1, L_2 of \mathcal{L} (with the notation in (1)). By definition, $p \in \overline{L_1}^A \cap \overline{L_2}^A$ and there exists a ball $\mathbb{B}(p, 2\varepsilon) \subset A$ such that $L_i \cap \mathbb{B}(p, 2\varepsilon)$ is closed in $\mathbb{B}(p, 2\varepsilon) - \mathcal{S}$, $i = 1, 2$. Since \mathcal{S} is countable, we may assume that the sphere $\mathbb{S}^2(p, \varepsilon)$ is disjoint from \mathcal{S} and intersects transversely $L_1 \cup L_2$. Next define $L_i(\varepsilon) = L_i \cap \overline{\mathbb{B}(p, \varepsilon)}$, $i = 1, 2$. Then $L_1(\varepsilon), L_2(\varepsilon)$ are disjoint, properly embedded minimal surfaces in $\overline{\mathbb{B}(p, \varepsilon)} - \mathcal{S}$. We will obtain a contradiction after replacing each $L_i(\varepsilon)$ by a component of it having p in its closure (we will use the same notation $L_i(\varepsilon)$ for this component); hence we will assume from now on that $L_i(\varepsilon)$ is connected, $i = 1, 2$. Since \mathcal{S} is countable and closed, $\overline{\mathbb{B}(p, \varepsilon)} - \mathcal{S}$ is a simply connected three-manifold with boundary. Hence, $\overline{L_1(\varepsilon)}$ separates $\overline{\mathbb{B}(p, \varepsilon)}$ into two connected components, and the same holds for $\overline{L_2(\varepsilon)}$. Let N be the closure of the component of $\overline{\mathbb{B}(p, \varepsilon)} - (\overline{L_1(\varepsilon)} \cup \overline{L_2(\varepsilon)})$ that contains both $\overline{L_1(\varepsilon)}, \overline{L_2(\varepsilon)}$ in its boundary.

Using ∂N as a barrier for solving Plateau problems in N , then from a compact exhaustion of $\overline{L_1(\varepsilon)} - \mathcal{S}$, we produce a properly embedded, area-minimizing varifold $\Sigma_1 \subset N - \mathcal{S}$ with $\partial \Sigma_1 = \partial L_1(\varepsilon)$ and $p \in \overline{\Sigma_1}$ and that separates $\overline{L_1(\varepsilon)}$ from $\overline{L_2(\varepsilon)}$ (see Meeks and Yau [35] for similar construction and a description of this barrier type construction). By regularity properties of area-minimizing varifolds, $\overline{\Sigma_1}$ is regular except possibly at points in $\overline{\Sigma_1} \cap \mathcal{S}$. Now consider $\overline{\Sigma_1} - \mathcal{S}$ to lie in $\overline{\mathbb{B}(p, \varepsilon)} - \mathcal{S}$ and so, $\overline{\Sigma_1} - \mathcal{S}$ represents a minimal lamination of $\overline{\mathbb{B}(p, \varepsilon)} - \mathcal{S}$ with stable leaves. Since \mathcal{S} is closed and countable, Corollary 2.5 implies that Σ_1 extends smoothly across $\mathcal{S} \cap \overline{\mathbb{B}(p, \varepsilon)}$. Exchanging $\overline{L_1(\varepsilon)}$ by $\overline{\Sigma_1}$ and reasoning analogously, we find an embedded, area-minimizing surface Σ_2 between $\overline{\Sigma_1}$ and $\overline{L_2(\varepsilon)}$, with $\partial \Sigma_2 = \partial L_2(\varepsilon)$, such that $p \in \overline{\Sigma_2}$ and

$\bar{\Sigma}_2$ is smooth. Clearly $\bar{\Sigma}_1, \bar{\Sigma}_2$ contradict the interior maximum principle at p , which proves the proposition. \square

Using similar arguments, we can extend Proposition 2.6 to the case of a general Riemannian three-manifold (for the proof to work and using the same notation as above, we also need the part of the boundary of N coming from the boundary of $\mathbb{B}(p, \varepsilon)$ to have positive mean curvature, which can be assumed by choosing ε small enough). The following result is a consequence of this generalization.

Corollary 2.7 *Let $\bar{B}_N(p, R)$ be a compact ball centered at a point p in a Riemannian three-manifold N , with radius $R > 0$. Suppose $M_1, M_2 \subset \bar{B}_N(p, R) - \{p\}$ are two disjoint, properly embedded minimal surfaces with boundaries $\partial M_i \subset \partial B_N(p, R)$, $i = 1, 2$. Then, at most one of these surfaces is noncompact, and in this case the noncompact component has just one end. Furthermore, if M is a properly embedded, smooth minimal surface of finite genus in $\bar{B}_N(p, R) - \{p\}$ with $\partial M \subset \partial B_N(p, R)$, then its closure \bar{M} is a smooth, compact, embedded minimal surface in $\bar{B}_N(p, R)$.*

Proof. Let M_1, M_2 be surfaces as in the statement of the corollary. If both surfaces are noncompact, then $\bar{M}_1 \cup \bar{M}_2$ forms a singular minimal lamination \mathcal{L} of $\bar{B}_N(p, R)$ with p as the only singular point of \mathcal{L} . By definition, p is a singular leaf point of both M_1 and M_2 , which contradicts Proposition 2.6 in the general Riemannian manifold setting (note that it suffices to find a contradiction in a sufficiently small ball centered at p , hence we can assume convexity for the boundary of this smaller ball). By applying the same argument in a smaller ball centered at p , we deduce directly that if M_1 is not compact, then M_1 has only one end.

Finally, consider a properly embedded, smooth minimal surface M of finite genus in $\bar{B}_N(p, R) - \{p\}$ with $\partial M \subset \partial B_N(p, R)$, and suppose that M is noncompact. We may assume, by passing to a smaller $R > 0$ and using the arguments in the previous paragraph, that M has just one end and that M is an annulus. We can also assume that the exponential map \exp_p yields \mathbb{R}^3 -coordinates on $B_N(p, r)$ centered at $p \equiv \vec{0}$, for $r > 0$ small enough. Since M is a locally rectifiable 2-dimensional varifold with bounded (actually zero) mean curvature, Theorem 3.1 in Harvey and Lawson [15] implies that M has finite area. Under this finiteness condition, Allard proved ([1], Section 6.5) the existence of minimal limit tangent cones of M in \mathbb{R}^3 at the origin after homothetic rescaling of coordinates. By Corollary 5.1(3) of [1], M satisfies a monotonicity formula for the extrinsic area (even in this Riemannian setting, see Remark 4.4 in [1]), valid for surfaces with bounded mean curvature. In the present setting that M has mean curvature zero, Allard's monotonicity formula implies that with respect to the metric g_M on M induced by the ambient metric g on N , (M, g_M) has at most quadratic extrinsic area growth, in the sense that

$$r \in (0, R) \mapsto r^{-2} \text{Area}(M \cap \bar{B}_N(p, r), g_M) \quad \text{is bounded.}$$

Now consider the ambient conformal change of metric $g_1 = \frac{1}{d^2}g$, where d denotes the distance function in (N, g) to p . Then, $(M, g_1|_M) \subset (\bar{B}_N(p, R) - \{p\}, g_1)$ is a complete annulus with linear area growth and compact boundary. Such a surface is conformally a punctured disk \mathbb{D}^* (see Grigor'yan [13]). Thus, the related conformal harmonic map of \mathbb{D}^* extends to a harmonic map on the whole disk \mathbb{D} , that gives rise to a conformal, branched minimal immersion defined on \mathbb{D}

(see e.g., Grüter [14]). Since M is embedded near p , then p cannot be a branch point; hence M extends across p to a smooth, compact, embedded minimal surface. This finishes the proof of the corollary. \square

3 The proof of Theorem 1.4.

Suppose W is a closed countable subset of \mathbb{R}^3 and $\{M_n\}_n$ is a sequence of embedded minimal surfaces (possibly with boundary) in $A = \mathbb{R}^3 - W$, such that $\{M_n\}_n$ has locally positive injectivity radius in A . We will first produce the possibly singular limit lamination $\overline{\mathcal{L}}^A$ that appears in Theorem 1.4. If the M_n have uniformly locally bounded curvature in A , then it is a standard fact that a subsequence of the M_n converges to a minimal lamination \mathcal{L} of A with empty singular set and empty singular set of convergence (see for instance the arguments in the proof of Lemma 1.1 in Meeks and Rosenberg [32]). In this case, $\overline{\mathcal{L}}^A = \mathcal{L}$ and $S^A = \emptyset$. Otherwise, there exists a point $p \in A$ such that, after replacing by a subsequence, the supremum of the absolute Gaussian curvature of $M_n \cap \mathbb{B}(p, 1/k)$ diverges to ∞ as $n \rightarrow \infty$, for any k fixed. Since A is open, we can assume $\mathbb{B}(p, 1/k) \subset A$ for k large and thus, Proposition 1.1 in [8] (see also Theorem 13 in [33]) implies that the sequence of surfaces $\{M_n \cap \mathbb{B}(p, 1/k)\}_n$ is locally simply connected in $\mathbb{B}(p, 1/k)$. We will next describe both the limit object of the surfaces $M_n \cap \mathbb{B}(p, 1/k)$ as $n \rightarrow \infty$ and the surfaces themselves for n large; this description relies on Colding-Minicozzi theory and is adapted from a similar description in [23]; we have include it here as well for the sake of completeness.

(D) For k and n large, $M_n \cap \overline{\mathbb{B}}(p, 1/k)$ consists of compact disks with boundaries in $\mathbb{S}^2(p, 1/k)$. By Theorem 5.8 in [5], after a rotation of \mathbb{R}^3 and extracting a subsequence, each of the disks $M_n \cap \overline{\mathbb{B}}(p, 1/k)$ contains a 2-valued minimal graph⁴ defined on an annulus $\{(x_1, x_2, 0) \mid r_n^2 \leq x_1^2 + x_2^2 \leq R^2\}$ with inner radius $r_n \searrow 0$, for certain $R \in (r_n, 1/k)$ small but fixed. By the one-sided curvature estimates and other results in [7], for some k_0 sufficiently large, a subsequence of the surfaces $\{M_n \cap \mathbb{B}(p, 1/k_0)\}_n$ (denoted with the same indexes n) converges to a possibly singular minimal lamination $\overline{\mathcal{L}}_p$ of $\mathbb{B}(p, 1/k_0)$ with singular set $\mathcal{S}_p \subset \mathbb{B}(p, 1/k_0)$, related (regular) minimal lamination $\mathcal{L}_p \subset \mathbb{B}(p, 1/k_0) - \mathcal{S}_p$ and singular set of convergence $S(\mathcal{L}_p) \subset \mathcal{L}_p$. Moreover, \mathcal{L}_p contains a limit leaf with p in its closure, that is either a stable minimal disk $D(p)$ (if $p \in S(\mathcal{L}_p)$) or a stable punctured minimal disk $D(p, *)$ (if $p \in \mathcal{S}_p$), and in this last case $D(p, *)$ extends smoothly across p to a stable minimal disk $D(p)$ that is a leaf of $\overline{\mathcal{L}}_p$; this is Lemma II.2.3 in [9]. In fact, $D(p)$ appears as a limit of the previously mentioned 2-valued minimal graphs inside the M_n , that collapse into it. In both cases, the boundary of $D(p)$ is contained in $\mathbb{S}^2(p, 1/k_0)$ and $D(p) \cap \mathcal{S}_p \subseteq \{p\}$. By Corollary I.1.9 in [7], there is a solid double cone⁵ $\mathcal{C}_p \subset \mathbb{B}(p, 1/k_0)$ with vertex at p and axis orthogonal to the tangent plane $T_p D(p)$, that intersects $D(p)$ only at the point p and such that the complement of \mathcal{C}_p

⁴In polar coordinates (ρ, θ) on $\mathbb{R}^2 - \{0\}$ with $\rho > 0$ and $\theta \in \mathbb{R}$, a k -valued graph on an annulus of inner radius $r > 0$ and outer radius $R > r$, is a single-valued graph of a function $u(\rho, \theta)$ defined over $\{(\rho, \theta) \mid r \leq \rho \leq R, |\theta| \leq k\pi\}$, k being a positive integer.

⁵A solid double cone in \mathbb{R}^3 is a set that after a rotation and a translation, can be written as $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq \delta^{-2} x_3^2\}$ for some $\delta > 0$. A solid double cone in a ball is the intersection of a solid double cone with a ball centered at its vertex.

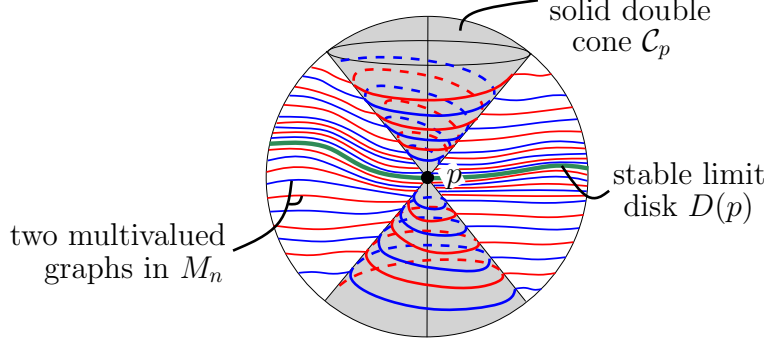


Figure 2: The local picture of disk-type portions of M_n around an isolated point $p \in S$. The stable minimal punctured disk $D(p, *)$ appears in the limit lamination \mathcal{L}_p , and extends smoothly through p to a stable minimal disk $D(p)$ that is orthogonal at p to the axis of the solid double cone \mathcal{C}_p .

in $\mathbb{B}(p, 1/k_0)$ does not intersect \mathcal{S}_p . Also, Colding-Minicozzi theory implies that for n large, $M_n \cap \mathbb{B}(p, 1/k_0)$ has the appearance outside \mathcal{C}_p of two highly-sheeted multivalued graphs over $D(p, *)$, see Figure 2. Furthermore:

- (D1) If $p \in S(\mathcal{L}_p)$ (in particular, $\overline{\mathcal{L}_p} = \mathcal{L}_p$ admits a local lamination structure around p), then after possibly choosing a larger k_0 , there exists a neighborhood of p in $\overline{\mathbb{B}(p, 1/k_0)}$ that is foliated by compact disks in \mathcal{L}_p , and $S(\mathcal{L}_p)$ intersects this family of disks transversely in a connected Lipschitz arc. This case corresponds to case (P) described in Section II.2 of [9]. In fact, the Lipschitz curve $S(\mathcal{L}_p)$ around p is a $C^{1,1}$ -curve orthogonal to the local foliation (Meeks [18, 19]), see Figure 3 left.
- (D2) If $p \in \mathcal{S}_p$, then after possibly passing to a larger k_0 , a subsequence of the surfaces $\{M_n \cap \mathbb{B}(p, 1/k_0)\}_n$ (denoted with the same indexes n) converges C^α , $\alpha \in (0, 1)$, on compact subsets of $\mathbb{B}(p, 1/k_0) - [\mathcal{S}_p \cup S(\mathcal{L}_p)]$ to the (regular) lamination $\mathcal{L}_p - S(\mathcal{L}_p)$.

To continue with the local description of case (D2), it is worth distinguishing two subcases:

- (D2-A) If p is an isolated point in \mathcal{S}_p , then the limit leaf $D(p, *)$ of \mathcal{L}_p is either the limit of two pairs of multivalued graphical leaves in \mathcal{L}_p (one pair on each side of $D(p, *)$), or $D(p, *)$ is the limit on one side of just one pair of multivalued graphical leaves in \mathcal{L}_p ; in this last case, p is the end point of an open arc $\Gamma \subset S(\mathcal{L}_p) \cap \mathcal{C}_p$, and a neighborhood of p in the closure of the component of $\mathbb{B}(p, 1/k_0) - D(p, *)$ that contains Γ is entirely foliated by disk leaves of \mathcal{L}_p , see Figure 3 center.
- (D2-B) p is not isolated as a point in \mathcal{S}_p . In this case, p is the limit of a sequence $\{p_m\}_m \subset \mathcal{S}_p \cap \mathcal{C}_p$. In particular, $D(p)$ is the limit of the related sequence of stable minimal disks $D(p_m)$, and $D(p, *)$ is the limit of a sequence of pairs of multivalued graphical leaves of $\mathcal{L}_p \cap [\mathbb{B}(p, 1/k_0) - (\mathcal{C}_p \cup \{D(p_m)\}_m)]$. Note that these singular points p_m might be isolated or not in \mathcal{S}_p , see Figure 3 right.

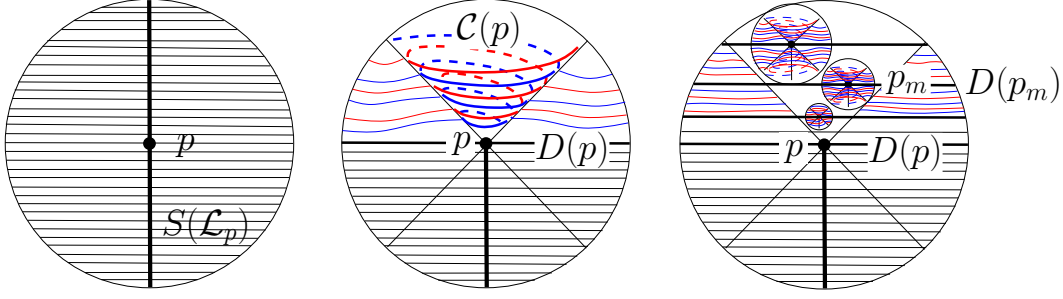


Figure 3: Left: Case (D1), in a neighborhood of a point $p \in S(\mathcal{L}_p)$. Center: Case (D2-A) for an isolated point $p \in \mathcal{S}_p$. In the picture, p is the end point of an arc contained in $S(\mathcal{L}_p)$, although $D(p, *)$ could also be the limit of two pairs of multivalued graphical leaves, one pair on each side. Right: Case (D2-B) for a nonisolated point $p \in \mathcal{S}_p$.

A standard diagonal argument implies, after extracting a subsequence, that the sequence $\{M_n\}_n$ converges to a possibly singular minimal lamination $\bar{\mathcal{L}}^A = \mathcal{L} \cup \mathcal{S}^A$ of A , with related (regular) lamination \mathcal{L} of $A - \mathcal{S}^A$, singular set $\mathcal{S}^A \subset A$ and with singular set of convergence $S(\mathcal{L}) \subset A - \mathcal{S}^A$ of the M_n to \mathcal{L} . Furthermore, in a neighborhood of every point $p \in \mathcal{S}^A \cup S(\mathcal{L})$, $\bar{\mathcal{L}}^A$ has the appearance of the possibly singular minimal lamination $\bar{\mathcal{L}}_p$ described above. Note that when $p \in S(\mathcal{L})$, since $S(\mathcal{L}) \subset \mathcal{L}$, then $\bar{\mathcal{L}}_p$ is a regular minimal lamination that must be a foliation near p .

Next we describe the structure of the closure $\bar{\mathcal{L}}$ of \mathcal{L} in \mathbb{R}^3 . $\bar{\mathcal{L}}$ can be written as

$$\bar{\mathcal{L}} = \bar{\mathcal{L}}^A \cup (W \cap \bar{\mathcal{L}}) = (\mathcal{L} \cup \mathcal{S}^A) \cup (W \cap \bar{\mathcal{L}})^{\text{lam}} \cup (W \cap \bar{\mathcal{L}})^{\text{sing}}, \quad (2)$$

(all unions in (2) are disjoint), where

$$\begin{aligned} (W \cap \bar{\mathcal{L}})^{\text{sing}} &= \{p \in W \cap \bar{\mathcal{L}} \mid \bar{\mathcal{L}} \text{ does not admit locally a lamination structure around } p\} \\ (W \cap \bar{\mathcal{L}})^{\text{lam}} &= (W \cap \bar{\mathcal{L}}) - (W \cap \bar{\mathcal{L}})^{\text{sing}}. \end{aligned}$$

Consider the set

$$\mathcal{S} = \mathcal{S}^A \cup (W \cap \bar{\mathcal{L}})^{\text{sing}}, \quad (3)$$

that is closed in \mathbb{R}^3 . If we define

$$\mathcal{L}_1 = \mathcal{L} \cup (W \cap \bar{\mathcal{L}})^{\text{lam}}, \quad (4)$$

then \mathcal{L}_1 can be endowed naturally with a structure of a (regular) minimal lamination of the open set $\mathbb{R}^3 - \mathcal{S}$. Thus, the decomposition (2) gives that $\bar{\mathcal{L}}$ is a possibly singular lamination of \mathbb{R}^3 , with singular set \mathcal{S} and related (regular) lamination \mathcal{L}_1 , and so, to finish this section it remains to prove items 1, ..., 7 in the statement of Theorem 1.4.

Lemma 3.1 *Items 1, 2 of Theorem 1.4 hold.*

Proof. Item 1 holds since the limit of a convergent sequence of planes is a plane. Next we show that the first sentence in item 2 holds.

Consider a limit leaf $\overline{\mathcal{L}}(L_1) = L_1 \cup \mathcal{S}_{L_1}$ of $\overline{\mathcal{L}}$, where L_1 is its related leaf of the regular lamination \mathcal{L}_1 and \mathcal{S}_{L_1} is the set of singular leaf points of L_1 . Thus, L_1 is a limit leaf of \mathcal{L}_1 . As \mathcal{L}_1 is a regular lamination of $\mathbb{R}^3 - \mathcal{S}$, then the stable limit leaf theorem [26, 27] applies in this case and gives that the two-sided cover of L_1 is stable. Since the set $\text{Lim}(\mathcal{L}_1)$ of limit leaves of \mathcal{L}_1 forms a sublamination (closeness of $\text{Lim}(\mathcal{L}_1)$ follows essentially from taking double limits), then the first sentence of item 2 of Theorem 1.4 reduces to checking that $\overline{\mathcal{L}}(L_1)$ is a plane. To do this, we will distinguish two cases.

(C1) If the M_n have uniformly locally bounded Gaussian curvature in A , then $\overline{\mathcal{L}}^A$ is a (regular) minimal lamination of A , i.e., $\overline{\mathcal{L}}^A = \mathcal{L}$ and $\mathcal{S}^A = \emptyset$. Hence, $\mathcal{S} \subset W$ and thus, \mathcal{S} is a closed countable set of \mathbb{R}^3 . Applying Corollary 2.5 we deduce that L_1 extends across $\overline{L_1} \cap W$ and its two-sided cover is a stable minimal surface. Since such an extension is clearly complete, it follows that the extension of L_1 across $\overline{L_1} \cap W$ is a plane. But this extension coincides with $\overline{\mathcal{L}}(L_1)$ and we are done in this case.

(C2) Suppose now that the M_n do not have uniformly locally bounded Gaussian curvature in A . By construction, we can decompose $L_1 = L_A \cup [L_1 \cap (W \cap \overline{\mathcal{L}})]^{\text{lam}}$, where L_A is a leaf of the (regular) minimal lamination $\mathcal{L} = \overline{\mathcal{L}}^A - \mathcal{S}^A$ of $A - \mathcal{S}^A$. Note that the two-sided cover of L_A is stable, since the same holds for L_1 by the stable limit leaf theorem [26, 27].

Consider the union \tilde{L}_A of L_A with all points $q \in \mathcal{S}^A$ such that the related punctured disk $D(q, *)$ defined in (D) above is contained in L_A . Clearly, \tilde{L}_A is a (smooth) minimal surface and the two-sided cover of \tilde{L}_A is stable. We claim that \tilde{L}_A is complete outside W in the sense that every divergent arc $\alpha: [0, 1) \rightarrow \tilde{L}_A$ of finite length has its limiting end point in W .

Arguing by contradiction, suppose that there exists a divergent arc $\alpha: [0, l) \rightarrow \tilde{L}_A$ of length l such that

$$q := \lim_{t \rightarrow l^-} \alpha(t) \in \overline{\tilde{L}_A} - W = \overline{L_A}^A = \overline{L_A} \cap A.$$

Therefore, there exists $\delta > 0$ such that $\alpha(t) \in \overline{\mathbb{B}}(q, \varepsilon)$ for every $t \in [l - \delta, l)$, where $\overline{\mathbb{B}}(q, \varepsilon)$ is the closed ball that appears in description (D) (with $\varepsilon = 1/k_0$), and $\varepsilon > 0$ is taken sufficiently small so that $\overline{\mathbb{B}}(q, \varepsilon) \subset A$. Note that by construction, $\alpha(t) \notin D(q, *)$ for every $t \in [l - \delta, l)$. As $D(q)$ separates $\overline{\mathbb{B}}(q, \varepsilon)$, then $\alpha([l - \delta, l))$ is contained in one of the two halfballs of $\overline{\mathbb{B}}(q, \varepsilon) - D(q)$, say in the upper “halfball” \mathbb{B}^+ (we can choose orthogonal coordinates in \mathbb{R}^3 centered at q so that $T_q D(q)$ is the (x_1, x_2) -plane). In particular, there cannot exist a sequence $\{q_m\}_m \subset \mathcal{S}^A$ converging to q in \mathbb{B}^+ , because otherwise q_m produces via (D2-B) a related disk $D(q_m)$ that is proper in \mathbb{B}^+ , such that the sequence $\{D(q_m)\}_m$ converges to $D(q)$ as $m \rightarrow \infty$; as $\alpha(l - \delta)$ lies above one of these disks $D(q_k)$ for k sufficiently large, then $\alpha([l - \delta, l))$ lies entirely above $D(q_k)$, which contradicts that γ limits to q . Therefore, after possibly choosing a smaller ε , we can assume that there are no points of \mathcal{S}^A in \mathbb{B}^+ other than q . Now consider the lamination \mathcal{L}' of $\overline{\mathbb{B}}(q, \varepsilon) - \{q\}$ given by $D(q, *)$ together with the closure of $L_A \cap \overline{\mathbb{B}}^+$ in $\overline{\mathbb{B}}(q, \varepsilon) - \{q\}$. As the leaves of \mathcal{L}' are all stable (if L_A is two-sided; otherwise we pass to a two-sided cover), then Corollary 2.5 implies that \mathcal{L}' extends smoothly across q , which is clearly impossible. This contradiction proves our claim that \tilde{L}_A is complete outside W .

Applying Corollary 7.2 in [31] to \tilde{L}_A , we deduce that closure of \tilde{L}_A in \mathbb{R}^3 is a plane, which finishes the proof of the first sentence in item 2 of Theorem 1.4.

As for the second sentence in item 2 of Theorem 1.4, take a leaf $L = \overline{\mathcal{L}}(L_1) = L_1 \cup \mathcal{S}_{L_1}$ of $\overline{\mathcal{L}}$ and suppose that p is a point in $A \cap \mathcal{S}_{L_1}$. As $\{M_n\}_n$ is locally simply connected outside W , then the description in (D)-(D1)-(D2) above implies that for $\varepsilon > 0$ sufficiently small, $L_1 \cap \overline{\mathbb{B}}(p, \varepsilon)$ equals the punctured disk $D(p, *)$ that appears in this description, since p is a singular leaf point of L_1 , also see Example 2.2-(B) in the Introduction. In particular, $L_1 \cap \overline{\mathbb{B}}(p, \varepsilon)$ is a limit leaf of the local lamination in $\mathbb{B}(p, \varepsilon)$ minus a certain solid cone centered at p . As L is connected, we conclude that L is a limit leaf of $\overline{\mathcal{L}}$. In this situation, the first sentence in item 2 of Theorem 1.4 implies that $L \in \mathcal{P}$. This finishes the proof of Lemma 3.1. \square

Next we prove item 4 of Theorem 1.4, since we shall made use of it in the proof of item 3 of the same theorem.

Lemma 3.2 *Item 4 of Theorem 1.4 holds.*

Proof. The local picture of $\overline{\mathcal{L}}^A$ described in (D)-(D1)-(D2) implies that through each point $q \in \mathcal{S}^A \cup \mathcal{S}(\mathcal{L})$ there passes a limit leaf of $\overline{\mathcal{L}}$ that, by item 2 of Theorem 1.4, must be a plane $P_q \in \text{Lim}(\overline{\mathcal{L}})$. Next we will prove that $P_q \cap (\mathcal{S}^A \cup \mathcal{S}(\mathcal{L}) \cup W)$ is a closed countable set. By the same local picture, we have that $P_q \cap (\mathcal{S}^A \cup \mathcal{S}(\mathcal{L}))$ is a discrete subset of $P_q - W$, that is clearly closed in the intrinsic topology of $P_q - W$. Thus the limit points of $P_q \cap (\mathcal{S}^A \cup \mathcal{S}(\mathcal{L}))$ lie in the closed countable set $P_q \cap W$. It then follows that $P_q \cap (\mathcal{S}^A \cup \mathcal{S}(\mathcal{L}) \cup W)$ is a closed countable set of \mathbb{R}^3 , and the lemma follows. \square

Lemma 3.3 *Item 3 of Theorem 1.4 holds.*

Proof. Suppose that P is a plane in $\mathcal{P} - \text{Lim}(\overline{\mathcal{L}})$. Since $\text{Lim}(\overline{\mathcal{L}})$ is a closed set of planes, we can choose $\delta > 0$ such that the 2δ -neighborhood of P is disjoint from $\text{Lim}(\overline{\mathcal{L}})$. By item 4 of Theorem 1.4, $\mathcal{S}(\mathcal{L}) \cup \mathcal{S}^A$ is at a positive distance at least 2δ from P .

If the δ -neighborhood $P(\delta)$ of P intersects $\overline{\mathcal{L}}$ in a portion of some leaf L' of $\overline{\mathcal{L}}$ different from P , then $L' \cap P(\delta)$, while it may have singularities in W , is proper as a set in $P(\delta)$: properness of the smooth surface $L' \cap [P(\delta) - W]$ is clear (as $P(2\delta)$ is disjoint from $\text{Lim}(\overline{\mathcal{L}})$); hence $L' \cap P(\delta)$ only intersects W in singular leaf points.

We now check that L' is disjoint from P . Arguing by contradiction, suppose that L' and P intersect. Note that every such intersection point q must lie in W by the maximum principle, and that q is a singular leaf point of both L' and P . This is impossible by Proposition 2.6 since W is countable. Therefore, L' does not intersect P . In this setting, we can use the proof of the halfspace theorem (Hoffman and Meeks [16]) with catenoid barriers (adapted to this situation with countably many singularities via Proposition 2.6) to obtain a contradiction to the existence of L' . Hence, $P(\delta) \cap \overline{\mathcal{L}} = P$, which proves the lemma. \square

Lemma 3.4 *Item 5 of Theorem 1.4 holds.*

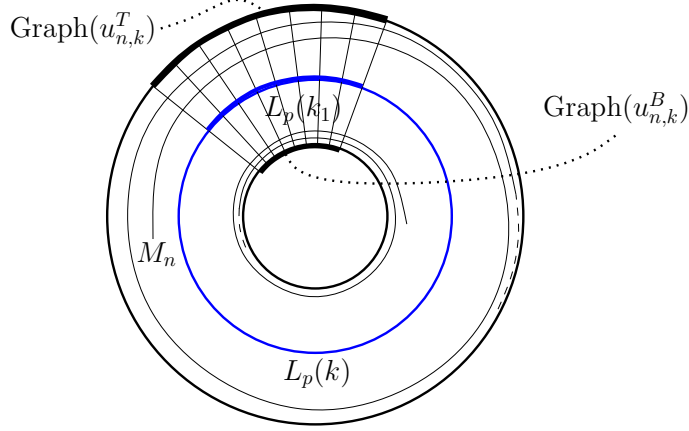


Figure 4: Schematic representation of the construction of a positive Jacobi function on $L_p(k) - \partial L_p(k)$ (we have simplified the figure by taking one dimension less); here $k_1 < k$, the components of $M_n \cap U(k_1)$ are normal graphs over $L_p(k_1) - \partial L_p(k_1)$ but the components of $M_n \cap U(k)$ are infinitely valued graphs over $L_p(k) - \partial L_p(k)$, created by holonomy. In any case, the closure $M_n \cap U(k)$ in $U(k)$ contains a “top” graphical leaf $\text{Graph}(u_{n,k}^T)$ and a “bottom” one $\text{Graph}(u_{n,k}^B)$ that are disjoint.

Proof. Suppose now that $p \in W \cap \bar{\mathcal{L}}$ satisfies one of the conditions of items 5.1, 5.2 in Theorem 1.4. First note that if p lies in the closure of $\mathcal{S}^A \cup S(\mathcal{L})$, then there passes a plane in \mathcal{P} through p by items 1 and 4 of Theorem 1.4 and we have the conclusion of item 5 in this case. Otherwise, we find an $R > 0$ such that the closed ball $\bar{\mathbb{B}}(p, R)$ does not intersect $\mathcal{S}^A \cup S(\mathcal{L})$. In particular, $\bar{\mathcal{L}} \cap [\bar{\mathbb{B}}(p, R) - W] = \mathcal{L} \cap [\bar{\mathbb{B}}(p, R) - W]$ and the surfaces M_n converge to \mathcal{L} on compact subsets of $\bar{\mathbb{B}}(p, R) - W$. Arguing by contradiction, suppose no plane in \mathcal{P} passes through p . Since \mathcal{P} is closed in \mathbb{R}^3 , then we can assume no plane in \mathcal{P} intersects $\bar{\mathbb{B}}(p, R)$, and hence item 2 of Theorem 1.4 implies that each leaf of $\bar{\mathcal{L}} \cap [\bar{\mathbb{B}}(p, R) - W]$ is proper in $\bar{\mathbb{B}}(p, R) - W$. By Proposition 2.6, the leaves of $\bar{\mathcal{L}} \cap \bar{\mathbb{B}}(p, R)$ are compact in $\bar{\mathbb{B}}(p, R)$ and pairwise disjoint.

Let L_p be the leaf of $\bar{\mathcal{L}} \cap \bar{\mathbb{B}}(p, R)$ that passes through p (p is a singular leaf point of the regular part of L_p , which in turn is contained in $L_p - W$). Note that the distance between L_p and the other leaves of $[\bar{\mathcal{L}} \cap \bar{\mathbb{B}}(p, R)] - L_p$ is positive, as follows also from Proposition 2.6 together with the fact that L_p is not a limit leaf. If 5.1 or 5.2 holds, then given a compact disk $D \subset L_p - W$ and $\varepsilon \in (0, R)$, there exists an integer $n_0 = n_0(D, \varepsilon)$ such that for $n \geq n_0$, there exist two pairwise disjoint disks D_1^n, D_2^n in M_n such that these disks are normal graphs over D with graphing functions f_1^n, f_2^n , respectively, each having norms less than ε . Observe that if we replace the disk D by a compact subdomain in $L_p - W$, then the graphing functions f_1^n, f_2^n might fail to be univalent. More precisely, consider a smooth⁶ compact exhaustion of $L_p - W$

$$L_p(1) \subset L_p(2) \subset \dots \subset L_p(k) \subset \dots \quad \text{with} \quad \partial L_p \subset \partial L_p(1).$$

⁶Smoothness of the compact exhaustion can be assumed as $\mathbb{S}^2(p, R)$ can be supposed to be transverse to L_p . Note that the topological boundary ∂L_p is nonempty and contained in $\mathbb{S}^2(p, R)$.

Fix $k \in \mathbb{N}$ large and consider the $r(k)$ -normal open regular neighborhood

$$U(k) = \{x + tN(x) \mid x \in L_p(k) - \partial L_p(k), |t| < r(k)\},$$

where N stands for the unit normal vector to L_p , and $r(k) \in (0, 1/k]$ is to be defined. For k sufficiently large, $U(k)$ is embedded in \mathbb{R}^3 for some $r(k) \in (0, 1/k]$. For $n \geq k$ sufficiently large, each component of $M_n \cap U(k)$ is either a normal graph or an infinitely valued graph over $L_p(k) - \partial L_p(k)$. In both cases, the validity of 5.1 or 5.2 implies that the closure of $M_n \cap U(k)$ in $U(k)$ contains two distinct leaves that are normal graphs over $L_p(k) - \partial L_p(k)$ with graphing functions $u_{n,k}^T, u_{n,k}^B : L_p(k) - \partial L_p(k) \rightarrow [-r(k), r(k)]$; see Figure 4. In order to obtain this description we are using that the surfaces $M_n \cap \mathbb{B}(p, R)$ have locally bounded Gaussian curvature in $\mathbb{B}(p, R) - W$, $\mathbb{B}(p, R) - W$ is simply connected and so $L_p - W$ is a two-sided minimal surface, and the fact that the leaf L_p is a positive distance from the other leaves of $\bar{\mathcal{L}} \cap \mathbb{B}(p, R)$. If we fix a point $p_0 \in L_p(k) - \partial L_p(k)$, then a subsequence of the positive functions

$$f_{n,k} = \frac{1}{(u_{n,k}^T - u_{n,k}^B)(p_0)} (u_{n,k}^T - u_{n,k}^B)$$

converges as $n \rightarrow \infty$ to a positive Jacobi function f on $L_p(k) - \partial L_p(k)$. This proves that $L_p(k) - \partial L_p(k)$ is stable for every k , which gives that $L_p - W$ is also stable. By Corollary 2.5 we deduce that $L_p - W$ extends across $W \cap \bar{\mathbb{B}}(p, R)$ to a smooth compact minimal surface that is L_p .

Let $L = L_1 \cap \mathcal{S}_{L_1}$ denote the leaf of $\bar{\mathcal{L}}$ that contains L_p , where L_1 is the (smooth) leaf of the regular part \mathcal{L}_1 of \mathcal{L} defined in (4), and \mathcal{S}_{L_1} is the set of singular leaf points of L_1 . As no plane in \mathcal{P} passes through p , then L_1 is not flat and so, L_1 is not a limit leaf of \mathcal{L}_1 by item 2 of Theorem 1.4. Let $\text{Lim}(L_1)$ be the set of limit points of L_1 . We claim that through every point $q \in \text{Lim}(L_1) \cap A$ there passes a plane that is contained in \bar{L} : If $q \in \mathcal{S}^A$, this follows from item 4 of Theorem 1.4; if on the contrary $q \in A - \mathcal{S}^A$, then the leaf L_2 of \mathcal{L}_1 that passes through q is a limit leaf of \mathcal{L}_1 , and thus, \bar{L}_2 is a plane by item 2 of Theorem 1.4. Now our claim holds. As through every point of $\text{Lim}(L_1) \cap A$ there passes a plane in \bar{L} , then a connectedness argument shows that L_1 is proper in $\Delta - W$, where $\Delta \subset \mathbb{R}^3$ is either an open halfspace or an open slab. As $\Delta - W$ is simply connected and L_1 is properly embedded in $\Delta - W$, then L_1 is orientable. Now consider a compact subdomain $\Omega \subset L_1$. As L_1 is not a limit leaf of \mathcal{L}_1 , then Ω is at a positive distance from any leaf of \mathcal{L}_1 different from L_1 . In particular, Ω admits a normal open neighborhood that is disjoint from any other leaf of \mathcal{L}_1 . In this setting, we can repeat the argument in the previous paragraph to construct a positive Jacobi function on Ω , which proves that Ω is stable. As Ω is any compact subdomain in L_1 , then we conclude that L_1 is stable as well.

We next prove that L_1 stays at a positive distance from every point $p_1 \in \mathcal{S}^A$: again arguing by contradiction, if this property fails to hold for a point $p_1 \in \mathcal{S}^A$, then portions of L_1 enter in every ball $\mathbb{B}(p_1, \varepsilon)$ of arbitrarily small radius. In this setting, the local description in (D)-(D1)-(D2) for a sufficiently small ball $\mathbb{B}(p_1, \varepsilon)$ implies that either L_1 contains the punctured disk $D(p_1, *)$ that appears in (D), or $L_1 \cap \mathbb{B}(p_1, \varepsilon)$ contains two multivalued graphs Σ that spiral together infinitely many times into $D(p_1, *)$ at one side of $D(p_1, *)$. The first possibility cannot occur as L_1 is not a limit leaf of \mathcal{L}_1 ; the second possibility cannot occur either, by Theorem 2.4 applied to the lamination $\Sigma \cup D(p_1, *)$ of $\mathbb{B}(p_1, \varepsilon) - \{p_1\}$, because Σ is stable. This contradiction shows that L_1 stays at a positive distance from every point $p_1 \in \mathcal{S}^A$.

Finally, as L_1 is a leaf of the lamination \mathcal{L}_1 of $\mathbb{R}^3 - \mathcal{S}$ (the singular set \mathcal{S} was defined in (3)) and L_1 stays at a positive distance from every point p_1 of \mathcal{S}^A , then we deduce that L_1 is complete outside W . As L_1 is stable, then Corollary 2.5 implies that L_1 extends across W to a complete stable minimal surface in \mathbb{R}^3 , hence a plane passing through p , which is absurd. Now the proof of the lemma is complete. \square

Proposition 3.5 *Item 6 of Theorem 1.4 holds.*

Proof. Let $L = L_1 \cup \mathcal{S}_{L_1}$ be a nonplanar leaf of $\bar{\mathcal{L}}$, where L_1 is the leaf of the regular part \mathcal{L}_1 of $\bar{\mathcal{L}}$ defined in (4) and \mathcal{S}_{L_1} is the set of singular leaf points of L_1 . As the argument to prove the proposition is delicate, we will organize it into four assertions.

Assertion 3.6 $L \cap (\mathcal{S}^A \cup \mathcal{S}(\mathcal{L})) = \emptyset$ and the convergence of portions of the M_n to L_1 is of multiplicity one.

Proof. If L intersects $\mathcal{S}(\mathcal{L})$ at a point x , then L is a smooth minimal surface around x . Since item 4 of Theorem 1.4 implies that there passes a plane $P_x \in \mathcal{P}$ through x , we conclude that $L = P_x$, which is impossible. If L intersects \mathcal{S}^A at a point y , then $y \in L \cap \mathcal{S} = \mathcal{S}_{L_1}$ where \mathcal{S} is the singular set of $\bar{\mathcal{L}}$ defined in (3). By item 4 of Theorem 1.4, there passes a plane $P_y \in \mathcal{P}$ through y , which implies that both L_1, P_y share the singular leaf point y . Since P_y intersects \mathcal{S} in a closed countable set (again by item 4 of Theorem 1.4), then Proposition 2.6 leads to a contradiction. Therefore, we have proved that $L \cap (\mathcal{S}^A \cup \mathcal{S}(\mathcal{L})) = \emptyset$. Finally, the property that the convergence of portions of the M_n to L_1 is of multiplicity one follows from the proof of Lemma 3.4. Now Assertion 3.6 follows. \square

To prove that either item 6.1 or 6.2 of Theorem 1.4 holds, we will distinguish two cases, depending on whether or not L is proper as a set in \mathbb{R}^3 .

(E1) SUPPOSE THAT L IS PROPER IN \mathbb{R}^3 .

Our goal is to show that item 6.1 of Theorem 1.4 holds. Since L is proper in \mathbb{R}^3 , all the points in $\mathcal{S} \cap \bar{L}$ are singular leaf points of L_1 , in particular $L = \bar{L}$. In this setting, the proof of the halfspace theorem that uses catenoid barriers together with Proposition 2.6 imply $\mathcal{P} = \emptyset$. By item 4 of Theorem 1.4, we have $\mathcal{S}^A \cup \mathcal{S}(\mathcal{L}) = \emptyset$. Thus, $\mathcal{S} \subset W$ by equality (3). To deduce item 6.1 of Theorem 1.4, it remains to prove that L is the unique leaf of $\bar{\mathcal{L}}$. Otherwise, $\bar{\mathcal{L}}$ contains a leaf $L' \neq L$, and L' is not flat since $\mathcal{P} = \emptyset$. Furthermore, L' is proper in \mathbb{R}^3 (if L' were nonproper then \bar{L}' would contain a limit leaf that is a plane in \mathcal{P}). Proposition 2.6 implies that L and L' do not intersect. The existence of the proper, possibly singular surfaces L, L' contradicts the proof of the strong halfspace theorem adapted to this singular setting via Proposition 2.6 (see [16, 34]), in which one first constructs a plane between L and L' and then applies the proof of the halfspace theorem. This proves that item 6.1 of Theorem 1.4 holds, as desired.

(E2) SUPPOSE THAT L IS NOT PROPER IN \mathbb{R}^3 .

In this second case we will demonstrate that item 6.2 of Theorem 1.4 holds, which will finish the proof of Proposition 3.5. As L is not proper in \mathbb{R}^3 , there exists a limit point q_0 of L , in the sense that there exists a sequence of points in L that converges to q_0 in \mathbb{R}^3 and that is intrinsically divergent in L . Therefore, L_1 is also nonproper in any extrinsic neighborhood of q_0 , which implies that q_0 is not a singular leaf point of L_1 and thus, q_0 is not contained in L .

Assertion 3.7 *Through any limit point q of L there passes a plane $P \in \mathcal{P}$. Furthermore, every point in such a plane P is a limit point of L .*

Proof. First note that such a limit point q of L cannot lie in L , by the discussion in the last paragraph. If q lies in the regular lamination \mathcal{L}_1 , then the leaf L'_1 of \mathcal{L}_1 that contains q is a limit leaf of \mathcal{L}_1 . By the arguments in the proof of Lemma 3.1, the closure $\overline{L'_1}$ must be a plane in \mathcal{P} , and the assertion holds in this case. Then, we may assume $q \in (\overline{L_1} \cap \mathcal{S}) - \mathcal{S}_{L_1}$. By Definition 1.1, this implies that for every open neighborhood V of q in \mathbb{R}^3 , then $L_1 \cap V$ fails to be closed in $V - \mathcal{S}$. Thus one can find a sequence $\{V_k\}_k$ of open neighborhoods of q and a sequence of points $x_k \in \overline{L_1 \cap V_k}^{V_k - \mathcal{S}} - (L_1 \cap V_k)$, $k \in \mathbb{N}$. Without loss of generality, we can assume $V_k \rightarrow \{q\}$ as $k \rightarrow \infty$. Fix $k \in \mathbb{N}$. Since x_k lies in the closure of $L_1 \cap V_k$ relative to $V_k - \mathcal{S}$, then there exists a sequence $\{y_k(m)\}_m \in L_1 \cap V_k$ with $y_k(m) \rightarrow x_k$ as $m \rightarrow \infty$. As $x_k \in (V_k - \mathcal{S}) - (L_1 \cap V_k)$, then $x_k \notin L_1$. Thus $\{y_k(m)\}_m$ converges to x_k in the topology of \mathbb{R}^3 but it does not converge to x_k in the intrinsic topology of L_1 (otherwise x_k would lie in L_1 since $x_k \notin \mathcal{S}$). This gives that $x_k \in \text{Lim}(L_1)$, and our previous arguments imply that there passes a plane in \mathcal{P} through x_k . Since this happens for all k , $x_k \rightarrow q$ as $k \rightarrow \infty$ and \mathcal{P} is a closed set of planes, then there also passes a plane in \mathcal{P} through q and the assertion is proved. \square

We continue with the proof of item 6.2 of Theorem 1.4 in case (E2). Since L is not proper in \mathbb{R}^3 and through any limit point of L there passes a plane in \mathcal{P} , a straightforward connectedness argument shows that $\overline{L} = L \cup \mathcal{P}(L)$ with $\mathcal{P}(L)$ consisting of one or two planes. In particular, L is proper in the component $C(L)$ of $\mathbb{R}^3 - \mathcal{P}(L)$ that contains L .

Assertion 3.8 *In the above situation, $C(L) \cap \overline{\mathcal{L}} = L$.*

Proof. Since L is connected and nonflat, there are no planar leaves of $\overline{\mathcal{L}}$ in $C(L)$. Reasoning by contradiction, suppose that L' is a nonflat leaf of $\overline{\mathcal{L}}$ that is different from L and that intersects $C(L)$. Since L and L' are proper in $C(L)$, the maximum principle together with Proposition 2.6 imply that $L \cap L' = \emptyset$. Reversing the roles of L and L' one can easily check that $\mathcal{P}(L) = \mathcal{P}(L')$ and $C(L) = C(L')$. As both $L - \mathcal{S}$ and $L' - \mathcal{S}$ are properly embedded smooth surfaces in the simply connected region $C(L) - \mathcal{S}$ (because $\mathcal{S} \cap C(L) \subset W$ is countable), then $L \cup L'$ bounds a closed region X in $C(L)$; since the two boundary components of X are good barriers for solving Plateau problems in X (in spite of being singular by using Proposition 2.6), a standard argument (see Meeks, Simon and Yau [34]) shows that there exists a properly embedded, least-area surface $\Sigma \subset X$ that separates L from L' in X , and hence separates L from L' in $C(L)$. However, since X is not necessarily complete (note that every divergent path in X with finite length must have a limit point in $\mathcal{S} \cap [L \cup L' \cup \mathcal{P}(L)]$), then the surface Σ might fail to be complete. On the other hand, Assertion 3.6 applied to L, L' implies that neither of the surfaces L, L' intersects $\mathcal{S}^A \cup \mathcal{S}(\mathcal{L})$, because both L, L' are not flat. This implies that $\mathcal{S} \cap (L \cup L') \subset W$; in particular, $\mathcal{S} \cap [L \cup L' \cup \mathcal{P}(L)]$ is closed and countable. As Σ , when considered to be a surface in \mathbb{R}^3 , is complete outside the closed countable $\mathcal{S} \cap [L \cup L' \cup \mathcal{P}(L)]$, then Corollary 2.5 implies that Σ extends to a complete, stable minimal surface $\overline{\Sigma}$ in \mathbb{R}^3 . Therefore, $\overline{\Sigma}$ is a plane. This is impossible as $\mathcal{P}(L) = \mathcal{P}(L')$ but L and L' lie on opposite sides of a plane. This proves the assertion. \square

Assertion 3.9 *Every open ε -neighborhood $P(\varepsilon)$ of a plane $P \in \mathcal{P}(L)$ intersects the surface L_1 in a connected smooth surface with unbounded Gaussian curvature.*

Proof. After a rotation, we may assume that $P = \{x_3 = 0\}$ and L limits to P from above P . Given $\varepsilon > 0$ small enough so that $\{0 < x_3 \leq \varepsilon\} \subset C(L)$, we consider the smooth minimal surface

$$L_1(\varepsilon) = L_1 \cap \{0 < x_3 \leq \varepsilon\}. \quad (5)$$

Note that $L_1(\varepsilon)$ is possibly incomplete (completeness of $L_1(\varepsilon)$ may fail in the set $\mathcal{S} \cap \{0 \leq x_3 \leq \varepsilon\}$). Since $\mathcal{S} \subset \mathcal{S}^A \cup W$, W is countable and $L \cap \mathcal{S}^A = \emptyset$ by item 4 of Theorem 1.4, then we may also choose ε so that the closure $\overline{L_1(\varepsilon)}$ in \mathbb{R}^3 of $L_1(\varepsilon)$ does not have singularities in the plane $\{x_3 = \varepsilon\}$. In a similar way as in the proof of Assertion 3.8, applying the proof of Theorem 1.6 in [32] and using the local extendability of a stable minimal surface in $\overline{C(L)}$ that is complete outside a closed countable set and has its boundary in a plane in $C(L)$, one sees that $\{0 \leq x_3 \leq \varepsilon\}$ intersects L in a connected set.

We next prove that the Gaussian curvature of $L_1(\varepsilon)$ is unbounded. Reasoning by contradiction, assume $L_1(\varepsilon)$ has bounded Gaussian curvature. In this case, $L_1(\varepsilon) \cup [P - (W \cap \overline{\mathcal{L}})^{\text{sing}}]$ is a relatively closed set of $\{-1 < x_3 < \varepsilon\} - (W \cap \overline{\mathcal{L}})^{\text{sing}}$ with bounded second fundamental form, hence $L_1(\varepsilon) \cup [P - (W \cap \overline{\mathcal{L}})^{\text{sing}}]$ is a minimal lamination of $\{-1 < x_3 < \varepsilon\} - (W \cap \overline{\mathcal{L}})^{\text{sing}}$. By Theorem 2.4, $L_1(\varepsilon) \cup P$ is a minimal lamination of $\{-1 < x_3 < \varepsilon\}$. In this situation with bounded Gaussian curvature, one can apply Lemma 1.4 in [32] to deduce that $L_1(\varepsilon)$ is a graph over its projection to P , in particular it is proper in the closed slab $\{0 \leq x_3 \leq \varepsilon\}$, which contradicts the proof of the Halfspace Theorem. Hence, $L_1(\varepsilon)$ has unbounded Gaussian curvature. \square

The main statements of item 6.2 and item 6.2(a) of Theorem 1.4 are now proven under the hypothesis of Case (E2); it remains to prove that the additional statements 6.2(b) and 6.2(c) hold to complete the proof of Proposition 3.5. This is a technical part of the proof, where the local picture theorem on the scale of topology [23] will play a crucial role.

Remark 3.10 In the first item of the next assertion, one can ask if it is the case that every open ε -neighborhood $P(\varepsilon)$ of P intersects the surface L_1 in a connected smooth surface with infinite genus, without making the additional assumption that the plane $P \in \mathcal{P}(L)$ contains a singularity of $\overline{\mathcal{L}}$. The answer to this question is unclear to the authors.

Assertion 3.11 *1. If a plane $P \in \mathcal{P}(L)$ contains a singularity of $\overline{\mathcal{L}}$, then every open ε -neighborhood $P(\varepsilon)$ of P intersects the surface L_1 in a connected smooth surface with infinite genus.*

2. The leaf L_1 has infinite genus.

Proof. We first prove that item 1 of the assertion implies item 2. Suppose that L_1 has finite genus and item 1 holds. Item 1 implies that each plane in $\mathcal{P}(L)$ contains no singularities of $\overline{\mathcal{L}}$. As L is proper in $C(L)$, then Corollary 2.7 implies that $\overline{L_1}$ has no singularities in $C(L)$ (to see this, observe that such a singularity q would belong to W , hence q could be assumed to be isolated in W by Baire's Theorem, and now Corollary 2.7 applies to give a contradiction). Hence, $\overline{L_1}$ is a minimal lamination of \mathbb{R}^3 whose leaves are the nonflat surface L_1 together with the nonempty set

of planes in $\mathcal{P}(L)$. The fact that L_1 has finite genus contradicts Corollary 1 in [24], which states that every nonplanar leaf of a minimal lamination of \mathbb{R}^3 with more than one leaf has infinite genus.

We next prove item 1 holds; this will complete the proof of the assertion and the proof of Proposition 3.5. Suppose that the (x_1, x_2) -plane $P \in \mathcal{P}(L)$ contains a singularity of \bar{L} . To finish the proof of Assertion 3.11, it remains to demonstrate that for every $\varepsilon > 0$, the surface $L_1(\varepsilon)$ given by (5) has infinite genus. If this infinite genus property were to fail, then we first choose ε sufficiently small so that $L_1(\varepsilon)$ has genus zero, keeping the property that $\overline{L_1(\varepsilon)}$ does not have singularities in the plane $\{x_3 = \varepsilon\}$. As L is proper in $C(L)$, then Corollary 2.7 implies that $\overline{L_1(\varepsilon)}$ has no singularities in $\{0 < x_3 \leq \varepsilon\}$. Thus, $L_1(\varepsilon)$ is a smooth, connected minimal surface with genus zero, that is complete outside a nonempty closed countable set

$$S' \subset S \cap P \tag{6}$$

(none of the points in S' can be a singular leaf point of L_1 , by Proposition 2.6), and the boundary of $L_1(\varepsilon)$ lies in the plane $\{x_3 = \varepsilon\}$. Since S' is nonempty, closed and countable, Baire's Theorem insures that there exists an isolated point $q \in S'$. After a translation and homothety, assume $q = \vec{0}$ and $S' \cap \mathbb{B}(2\delta) = \{\vec{0}\}$ for some positive $\delta < \frac{\varepsilon}{2}$.

Let I_{L_1} be the injectivity radius function of L_1 . We will find the desired contradiction by discussing the cases (E2-A), (E2-B) below, depending on whether or not $(I_{L_1})/|\cdot|$ is bounded away from zero in $L_1 \cap \mathbb{B}(\delta)$ ($|\cdot|$ denotes distance to the origin in \mathbb{R}^3).

(E2-A) SUPPOSE THAT $(I_{L_1})/|\cdot|$ IS NOT BOUNDED AWAY FROM ZERO IN $L_1 \cap \mathbb{B}(\delta)$.

We will use the local picture theorem on the scale of topology together with a flux argument to discard this case. By Theorem 1.1 in [23] (see also Remark 4.31 in the same paper), there exists a sequence of points $\{p_n\}_n \subset L_1$ called *points of almost-minimal injectivity radius*, such that the following properties hold:

$$(F1) \quad p_n \rightarrow \vec{0} \text{ and } \frac{I_{L_1}(p_n)}{|p_n|} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$(F2) \quad \text{For all } n \in \mathbb{N}, \text{ there exists } \varepsilon_n \in (0, |p_n|/2) \text{ such that the closure } L(n) \text{ of the component of } L_1(\varepsilon) \cap \mathbb{B}(p_n, \varepsilon_n) = L \cap \mathbb{B}(p_n, \varepsilon_n) \text{ that contains } p_n \text{ is compact and has its boundary in } \mathbb{S}^2(p_n, \varepsilon_n).$$

$$(F3) \quad \text{Defining } \lambda_n = 1/I_{L_1}(p_n) \in \mathbb{R}^+, \text{ then:}$$

$$(F3.1) \quad \text{The injectivity radius function of } L_1 \text{ restricted to } L(n), \text{ denoted by } I_{L(n)}, \text{ satisfies } \lambda_n I_{L(n)} \geq 1 - \frac{1}{n} \text{ on } L(n).$$

$$(F3.2) \quad \lambda_n \varepsilon_n \rightarrow \infty \text{ as } n \rightarrow \infty.$$

$$(F3.3) \quad \text{The sequence of surfaces } \{\widehat{L}(n) := \lambda_n[L(n) - p_n]\}_n \text{ converges as } n \rightarrow \infty \text{ to either a nonsimply connected, properly embedded minimal surface } \widehat{L}(\infty) \subset \mathbb{R}^3 \text{ of genus zero or to a minimal parking garage structure in } \mathbb{R}^3 \text{ with two oppositely oriented columns. By the classification of genus zero properly embedded minimal surfaces in } \mathbb{R}^3 \text{ [10, 17, 28], the surface } \widehat{L}(\infty) \text{ is either a catenoid or a Riemann minimal example if it occurs.}$$

We first consider the case where the limit object of $\{\widehat{L}(n)\}_n$ is a catenoid $\widehat{L}(\infty)$. Let Π be the plane in \mathbb{R}^3 that intersects $\widehat{L}(\infty)$ orthogonally along its waist circle Γ . Let $\Gamma_n \subset \widehat{L}(n) \cap \Pi$ be nearby simple closed planar curves in $\widehat{L}(n)$ for n large and let $\gamma_n = p_n + \frac{1}{\lambda_n} \Gamma_n$ be the related simple closed planar curves in $L(n)$. In particular, the sequence of simple closed curves $\gamma_n \subset L(n)$ near p_n have lengths converging to zero as $n \rightarrow \infty$, and when viewed to be sets, these curves converge to the origin $\vec{0}$. Note that by the convex hull property, the curves γ_n are not homologous to zero in $L_1(\varepsilon)$.

As $L_1(\varepsilon)$ has genus zero and the γ_n are not homologous to zero, then γ_n separates $L_1(\varepsilon)$ into two subdomains. Since $\{0 < x_3 \leq \varepsilon\}$ is simply connected and $L_1(\varepsilon)$ is properly embedded in $\{0 < x_3 \leq \varepsilon\}$, then $L_1(\varepsilon)$ separates $\{0 < x_3 \leq \varepsilon\}$ into two components. Let X_n be the closure of a component of $\{0 < x_3 \leq \varepsilon\} - L_1(\varepsilon)$ in which γ_n fails to bound a disk; after extracting a subsequence, we can assume that $X = X_n$ does not depend upon n . Our previous arguments using $L_1(\varepsilon)$ as a barrier (see e.g., the proof of Assertion 3.8) imply that γ_n is contained in the boundary of a connected, area-minimizing, noncompact, orientable, properly embedded minimal surface $\Sigma_n \subset X$ (possibly incomplete, as X might not be complete) with the remainder of its boundary contained in $\partial L_1(\varepsilon)$. As $L_1(\varepsilon)$ has no singularities in $\{0 < x_3 \leq \varepsilon\}$, then Σ_n is a surface with boundary in $\partial L_1(\varepsilon) \cup \gamma_n$ and it is complete in \mathbb{R}^3 outside of the closed countable set $\{x_3 = 0\} \cap \mathcal{S}$. Since Σ_n is stable, then Corollary 2.5 insures that Σ_n extends to a complete, orientable, stable minimal surface $\overline{\Sigma}_n \subset \{0 \leq x_3 \leq \varepsilon\}$ with boundary in $\partial L_1(\varepsilon) \cup \gamma_n$. By the maximum principle, $\overline{\Sigma}_n$ is disjoint from $P = \{x_3 = 0\}$ and so $\overline{\Sigma}_n = \Sigma_n$. By curvature estimates for stable minimal surfaces, the second fundamental form of Σ_n is bounded in a fixed sized neighborhood $V_n(P)$ of P (size depending on n), which implies that Σ_n is proper in \mathbb{R}^3 (properness of Σ_n follows from an application of Lemma 1.4 in [32] to each component of the intersection of Σ_n with a sufficiently small fixed size neighborhood of P contained in $V_n(P)$). Therefore, Σ_n is a parabolic surface by Theorem 3.1 in [11].

Now fix a point $p_0 \in \partial L_1(\varepsilon) \cap \{x_3 = \varepsilon\}$. The curve γ_n separates $L_1(\varepsilon)$ into two components and let $L_1(n, \varepsilon)$ be the component containing p_0 . Note that for some regular value $\eta \in (0, \varepsilon)$ of $x_3|_{L_1(\varepsilon)}$ so that $\varepsilon - \eta$ is sufficiently small, the component $L_1(n, \varepsilon, \eta)$ of $L_1(n, \varepsilon) \cap \{x_3 \leq \eta\}$ that contains γ_n must contain a boundary component $\partial \subset L_1(n, \varepsilon) \cap \{x_3 = \eta\}$ intrinsically close to p_0 and ∂ does not depend on n .

Suppose for the moment that

$$x_3|_{L_1(n, \varepsilon, \eta)} \geq \min(x_3|_{\gamma_n}). \quad (7)$$

Under the above hypothesis, $L_1(n, \varepsilon, \eta)$ is properly embedded in \mathbb{R}^3 and then Theorem 3.1 in [11] implies that $L_1(n, \varepsilon, \eta)$ is a parabolic surface. Since $L_1(n, \varepsilon, \eta)$ is parabolic, the arguments in the proof of Claim 4.19 in [23] show that the scalar flux of the intrinsic gradient ∇x_3 of x_3 on $L_1(n, \varepsilon, \eta)$ across $\gamma_n \subset \partial L_1(n, \varepsilon, \eta)$, given by

$$F(\nabla x_3, \gamma_n) = \int_{\gamma_n} \langle \nabla x_3, \nu \rangle$$

where ν is the inward pointing conormal to $L_1(n, \varepsilon, \eta)$ along its boundary, is bounded from below by the positive number

$$-F(\nabla x_3, \partial) = - \int_{\partial} \langle \nabla x_3, \nu \rangle.$$

But this conclusion is impossible since the lengths of γ_n are converging to zero as $n \rightarrow \infty$. Thus to find the desired contradiction in the case that (E2-A) holds with the limit object of $\{\widehat{L}(n)\}_n$ being a catenoid, it remains to show that (7) holds for n sufficiently large.

Next note that as the stable minimal surface Σ_n is parabolic, then

$$x_3|_{\Sigma_n} \geq \min(x_3|_{\gamma_n}). \quad (8)$$

Inequality (8) implies that the ends of the catenoid $\widehat{L}(\infty)$ are horizontal. Also, since Σ_n is parabolic and the scalar flux of the intrinsic gradient $\nabla^{\Sigma_n} x_3$ across $\gamma_n \subset \partial \Sigma_n$ is converging to zero as $n \rightarrow \infty$, then similar reasoning as in the previous paragraph implies that the scalar flux of $\nabla^{\Sigma_n} x_3$ across $\Sigma_n \cap \{x_3 = \eta\}$ is converging to zero as $n \rightarrow \infty$. By curvature estimates for the stable minimal surface Σ_n , we conclude:

(G0) The spherical image of the Gauss map of Σ_n along $\Sigma_n \cap \{x_3 = \eta\}$ is contained in arbitrarily small neighborhoods of the north or south pole of $\mathbb{S}^2(1)$ for n sufficiently large.

Arguing similarly with fluxes also we deduce that there exists a positive constant C depending only on curvature estimates for stable minimal surfaces such that for any point of Σ_n of intrinsic distance greater than $C \cdot \text{Length}(\gamma_n)$ from $\partial \Sigma_n$, the normal line to Σ_n must make an angle of less than $\frac{\pi}{4}$ with the horizontal.

Let $\widehat{X}(\infty)$ be the nonsimply connected component of $\mathbb{R}^3 - \widehat{L}(\infty)$, and let $X(n) \subset \overline{\mathbb{B}}(p_n, \varepsilon_n)$ be the related solid annular regions defined by the condition

$$\lambda_n[X(n) - p_n] \text{ converges to } \widehat{X}(\infty) \text{ as } n \rightarrow \infty.$$

By letting ε_n converge to zero sufficiently quickly and after replacing by a subsequence, we can also assume that the domains $\lambda_n[\partial X(n) - p_n] \cap \mathbb{B}(n)$ are annuli that can be expressed as normal graphs over their projections to $\widehat{L}(\infty)$ with the C^1 -norm of the graphing functions less than $\frac{1}{n}$ and so that $\lambda_n \varepsilon_n = n$.

Curvature estimates for the stable minimal surface Σ_n and the flat horizontal asymptotic geometry of the catenoid $\widehat{L}(\infty)$ imply there is a constant $R > 1$ such that, for n sufficiently large, the components of $\Sigma_n \cap \left[\mathbb{B}(p_n, \frac{n}{2\lambda_n}) - \mathbb{B}(p_n, \frac{R}{\lambda_n}) \right]$ are graphs over their orthogonal projections to the (x_1, x_2) -plane. Since these graphs are part of Σ_n which is an area-minimizing surface in $X(n)$, then there is only one such graph for n sufficiently large. Note that by picking R sufficiently large, the graph $\Sigma_n \cap \left[\mathbb{B}(p_n, \frac{n}{2\lambda_n}) - \mathbb{B}(p_n, \frac{R}{\lambda_n}) \right]$ can be assumed to have arbitrarily small gradient.

There exists a connected compact neighborhood $U(\gamma_n)$ of γ_n in $\Sigma_n \cap \mathbb{B}(p_n, \varepsilon_n)$ with two boundary components, γ_n, α_n , such that the component Ω_n of $\Sigma_n - [U(\gamma_n) \cup \{\eta \leq x_3 \leq \varepsilon\}]$ with boundary curve $\alpha_n := \partial \Omega_n \cap \partial U(\gamma_n)$ satisfies:

(G1) α_n can be chosen so that it corresponds to the inner boundary component of the annular graph $\Sigma_n \cap \left[\mathbb{B}(p_n, \frac{n}{2\lambda_n}) - \mathbb{B}(p_n, \frac{R}{\lambda_n}) \right]$.

(G2) After choosing R sufficiently large, then the Gauss map G_n of Ω_n along α_n is almost constant and equal to the vertical normal vector of one of the ends of $\widehat{L}(\infty)$.

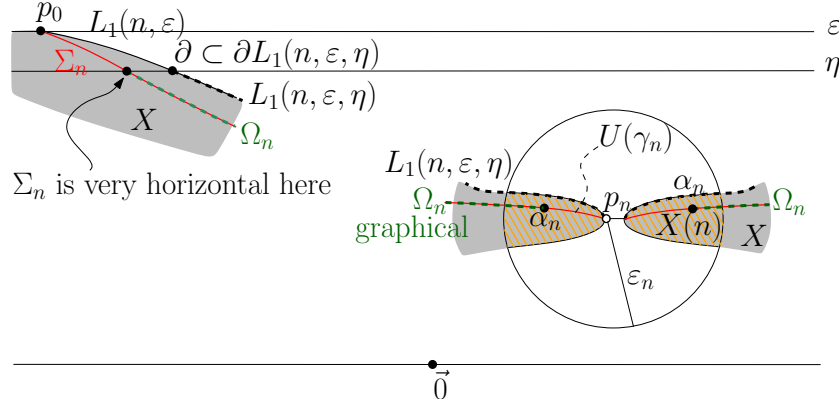


Figure 5: The graphical piece Ω_n inside the stable surface Σ_n together with a disk T_n (not represented in the figure) produce a piecewise smooth graph Y_n that separates the slab $\{0 \leq x_3 \leq \eta\}$, leaving the surface $L_1(n, \varepsilon, \eta)$ above Y_n .

(G3) For n sufficiently large, the intrinsic distance function $d_{\Sigma_n}(\cdot, \partial\Sigma_n)$ restricted to Ω_n is greater than $C \cdot \text{Length}(\gamma_n)$. In particular, the normal line to Ω_n must make an angle of less than $\frac{\pi}{4}$ with the horizontal.

Properties (G0) and (G2) together with the stability of Ω_n imply that $G_n(\Omega_n)$ can be assumed to be contained in an arbitrarily small neighborhood of the north or south pole in $\mathbb{S}^2(1)$, for n sufficiently large. In particular, the orthogonal projection from Ω_n to P is a proper submersion for n large, and its restriction to each boundary component of Ω_n is injective. In this situation, one can apply Lemma 1.4 in [32] to deduce that for n sufficiently large, Ω_n is a graph over its projection to P whose gradient has norm at most 1.

We next prove that for n sufficiently large, the curve α_n is the boundary of a small almost-horizontal disk $T_n \subset \mathbb{B}(p_n, \varepsilon_n)$ that intersects $L_1(n, \varepsilon, \eta)$ only along γ_n . The construction of this disk is clear from Figure 5 if, for n sufficiently large, $L_1 \cap \mathbb{B}(p_n, \frac{R}{\lambda_n})$ is the annulus $\partial X(n) \cap \mathbb{B}(p_n, \frac{R}{\lambda_n})$. Otherwise since $L_1 \cap \overline{\mathbb{B}}(p_n, \varepsilon_n)$ is compact, then $L_1 \cap \overline{\mathbb{B}}(p_n, \varepsilon_n)$ would contain a compact component Λ_n , disjoint from the annulus $\partial X(n)$, with its boundary in $\mathbb{S}^2(p_n, \varepsilon_n)$ and which intersects $\mathbb{B}(p_n, \frac{R}{\lambda_n})$. Then Λ_n could be used as a barrier to construct a compact stable minimal surface Λ'_n in $\mathbb{B}(p_n, \varepsilon_n) - L_1$ with boundary in $\mathbb{S}^2(p_n, \varepsilon_n)$ that intersects $\mathbb{B}(p_n, \frac{R}{\lambda_n})$. By fixing R and letting $n \rightarrow \infty$, the stable minimal surfaces Λ'_n can then be used to prove that the limit catenoid $\hat{L}(\infty)$ lies on one side of a complete stable minimal surface that is a plane, which is impossible. This shows that T_n exists.

The union Y_n of Ω_n with T_n is a graph over its projection to P and Y_n separates the slab $\{0 \leq x_3 \leq \eta\}$ into two components, one of whose closures contains the surface $L_1(n, \varepsilon, \eta)$. Since a subsequence of the graphs Ω_n converges to a minimal graph G contained in $\{0 \leq x_3 \leq \varepsilon - \eta\}$ that has $\vec{0}$ in its closure, then $G \cup \{\vec{0}\}$ is a smooth minimal surface, possibly with boundary. By the maximum principle for minimal surfaces, it now follows that the graphs Y_n converge as $n \rightarrow \infty$ to the entire plane P . Hence, for n sufficiently large, the boundary component ∂ of $L_1(n, \varepsilon, \eta)$ must lie in the region above the graph Y_n . This implies that the surface $L_1(n, \varepsilon, \eta)$ is properly

embedded in \mathbb{R}^3 since it lies above the proper graph Y_n (recall that L_1 is proper in $\{0 < x_3 \leq \varepsilon\}$). Hence by Theorem 3.1 in [11], $L_1(n, \varepsilon, \eta)$ is a parabolic surface and $x_3|_{L_1(n, \varepsilon, \eta)} \geq \min(x_3|_{\gamma_n})$. This completes the proof that (7) holds.

The above arguments show that the limit object $\widehat{L}(\infty)$ of the surfaces $\widehat{L}(n)$ defined in (F3.3) is not a catenoid. Thus, either the limit object is a Riemann minimal example or a minimal parking garage structure of \mathbb{R}^3 with two oppositely oriented columns. In either of these cases, there exist homotopically nontrivial closed curves $\tau_n \subset L(n)$ converging to $\vec{0}$ with lengths converging to zero that play the role of the waist curves γ_n of the forming catenoids in the previously considered case. In the case that the limit object is a Riemann minimal example, then the τ_n correspond to a circle of the limit Riemann minimal example and in the case that the limit object is a minimal parking garage structure, then the τ_n correspond to “connection loops” as described in item (B) of Proposition 4.20 in [23]. Using the closed curves τ_n in place of the curves γ_n , the arguments in the case where $\widehat{L}(\infty)$ was a catenoid can be adapted in a straightforward manner to obtain a contradiction. Thus, Case (E2-A) does not occur at any isolated point in S' .

(E2-B) SUPPOSE THAT THERE IS A CONSTANT $c > 0$ SUCH THAT $I_{L_1} \geq c |\cdot|$ IN $L_1 \cap \mathbb{B}(\delta)$. The contradiction in this case will be found after the application of the already proven parts of Theorem 1.4 to an appropriate sequence of homothetic expansions of $L_1(\varepsilon)$ from the origin. Since $S' \cap \mathbb{B}(2\delta) = \{\vec{0}\}$, we can apply Theorem 2.4 to the minimal lamination

$$\left(\overline{[L_1(\varepsilon) \cup P]} - \{\vec{0}\} \right) \cap \mathbb{B}(\delta)$$

of $\mathbb{B}(\delta) - \{\vec{0}\}$ to conclude that there exists a sequence of points $\{p_n\}_n \subset L_1(\varepsilon)$ converging to $\vec{0}$ such that $|K_{L_1}(p_n)|p_n|^2 \geq n$ for all n . Consider the sequence of embedded minimal surfaces with boundary

$$\widehat{L}'(n) = \frac{1}{|p_n|} [L_1(\varepsilon) \cap \mathbb{B}(\delta)],$$

all of which have genus zero. Note that the boundary of $\widehat{L}'(n)$ lies in the horizontal plane at height $\varepsilon/|p_n| \rightarrow \infty$ and that by our hypothesis in (E2-B), the injectivity radius function $I_{\widehat{L}'(n)}$ of $\widehat{L}'(n)$ satisfies

$$I_{\widehat{L}'(n)}(x) = \frac{1}{|p_n|} I_{L_1}(|p_n|x) \geq c|x|, \quad \text{for all } x \in \widehat{L}'(n). \quad (9)$$

Given $p \in \mathbb{R}^3$ and $\tau > 0$, consider the conical region

$$C^+(p, \tau) = \{(x_1, x_2, x_3) \mid (x_1 - x_1(p))^2 + (x_2 - x_2(p))^2 < \tau^{-2}(x_3 - x_3(p))^2\} \quad (10)$$

with vertex p and opening angle α with respect to the positive x_3 -axis such that $\cot \alpha = \tau$. A consequence of the scale invariant lower bound (9) for $I_{\widehat{L}'(n)}$ together with the intrinsic version of the one-sided curvature estimate by Colding-Minicozzi (Corollary 0.8 in [8]) is that

(H) Given $a > 0$ small, there exists $\tau > 0$ such that for every $r > 0$, $\widehat{L}'(n) - [\mathbb{B}(r) \cup C^+(\vec{0}, \tau)]$ has bounded Gaussian curvature on compact sets, with the bound independent of n , and for n sufficiently large, the components in this set consist of graphs and multivalued graphs over their projections to the (x_1, x_2) -plane P , with the norms of the gradients of the graphing functions being less than a .

In particular, the points $\frac{p_n}{|p_n|}$ lie in $\mathbb{S}^2(1) \cap C^+(\vec{0}, \tau)$ for n sufficiently large.

Another consequence of (9) is that the sequence $\{\widehat{L}'(n)\}_n$ has locally positive injectivity radius in the open set $B = \mathbb{R}^3 - \{\vec{0}\}$. Also observe that $\vec{0}$ lies in the closure of $\widehat{L}'(n)$ for all n . Applying the already proven conclusions of Theorem 1.4 before the list of items 1, \dots , 7 to the closed countable set $W = \{\vec{0}\}$ and to the sequence of minimal surfaces $\{\widehat{L}'(n)\}_n$, we conclude that there exists a (possibly empty) closed subset \mathcal{S}^B of $\mathbb{R}^3 - \{\vec{0}\}$ and a (regular) minimal lamination $\widehat{\mathcal{L}}$ of $\mathbb{R}^3 - [\{\vec{0}\} \cup \mathcal{S}^B]$ such that:

- (H1) After passing to a subsequence (denoted in the same way), $\{\widehat{L}'(n)\}_n$ converges C^α , $\alpha \in (0, 1)$, to $\widehat{\mathcal{L}}$ in $\mathbb{R}^3 - [\{\vec{0}\} \cup \mathcal{S}^B \cup S(\widehat{\mathcal{L}})]$, where $S(\widehat{\mathcal{L}}) \subset \widehat{\mathcal{L}}$ is the singular set of convergence of the $\widehat{L}'(n)$ to $\widehat{\mathcal{L}}$.
- (H2) The closure of $\widehat{\mathcal{L}}$ relative to B has the structure of a possibly singular minimal lamination of B with related regular lamination $\widehat{\mathcal{L}}$ and singular set \mathcal{S}^B .
- (H3) The closure $\overline{\widehat{\mathcal{L}}}$ of $\widehat{\mathcal{L}}$ in \mathbb{R}^3 has the structure of a possibly singular minimal lamination of \mathbb{R}^3 , whose singular set $\widehat{\mathcal{S}}$ satisfies $\widehat{\mathcal{S}} \subset \mathcal{S}^B \cup \{\vec{0}\}$.

Clearly, $\widehat{\mathcal{L}}, \mathcal{S}^B$ and $\widehat{\mathcal{S}}$ are contained in $\{x_3 \geq 0\}$. Since by construction the curvatures of the surfaces $\widehat{L}'(n)$ are unbounded on $\mathbb{S}^2(1)$, then $\mathcal{S}^B \cup S(\widehat{\mathcal{L}}) \neq \emptyset$.

By property (H) above, we deduce that $\mathcal{S}^B \cup S(\widehat{\mathcal{L}})$ lies in $C^+(\vec{0}, \tau)$. As the absolute Gaussian curvature of the $\widehat{L}'(n)$ at $\frac{p_n}{|p_n|}$ is at least n , we also deduce that $\mathcal{S}^B \cup S(\widehat{\mathcal{L}})$ intersects $\mathbb{S}^2(1)$ at some point $x_0 \in C^+(\vec{0}, \tau)$.

Claim 3.12 $\overline{\widehat{\mathcal{L}}}$ is the foliation of the closed upper halfspace of \mathbb{R}^3 by horizontal planes (in particular, $\widehat{\mathcal{S}} = \mathcal{S}^B = \emptyset$), and $S(\widehat{\mathcal{L}})$ is the positive x_3 -axis.

Proof. Observe that we cannot apply item 7 of Theorem 1.4 since it has not been proved yet. Instead, we argue as follows. Suppose for the moment that $\overline{\widehat{\mathcal{L}}}$ does not contain nonflat leaves. Then, the leaves of $\widehat{\mathcal{L}}$ are horizontal planes and $\widehat{\mathcal{S}} = \mathcal{S}^B = \emptyset$. Since $x_0 \in [\mathcal{S}^B \cup S(\widehat{\mathcal{L}})] \cap \mathbb{S}^2(1)$ and the sequence $\{\widehat{L}'(n)\}_n \subset \{x_3 > 0\}$ is locally simply connected in $\mathbb{R}^3 - \{\vec{0}\}$, then it follows from Corollary 0.8 in [8] and from Meeks' $C^{1,1}$ -regularity theorem [18] that $S(\widehat{\mathcal{L}}) = \{(0, 0, x_3) \mid x_3 > 0\}$. Thus, in order to finish the proof of Claim 3.12 we will suppose that $\overline{\widehat{\mathcal{L}}}$ contains a nonflat leaf L' and we will find a contradiction.

By definition of leaf of a singular lamination, we can decompose $L' = L'_1 \cup \mathcal{S}_{L'_1}$ where L'_1 is a leaf of the regular lamination $\widehat{\mathcal{L}}$ and $\mathcal{S}_{L'_1}$ is the set of singular leaf points of L'_1 . Note that L'_1 is not flat, and so the convergence of portions of the $\widehat{L}'(n)$ to L'_1 is of multiplicity one (see item 5 of Theorem 1.4). As the $\widehat{L}'(n)$ have genus zero, then the same holds for L'_1 .

Since $x_0 \in \mathcal{S}^B \cup S(\widehat{\mathcal{L}})$, then item 4 of Theorem 1.4 implies that there passes a plane P_{x_0} through x_0 , such that P_{x_0} is a leaf of $\overline{\widehat{\mathcal{L}}}$. Recall that we have also proven item 6 of Theorem 1.4 except for the property of $L'_1(\varepsilon') = L'_1 \cap \{0 < x_3 \leq \varepsilon'\}$ having infinite genus for every $\varepsilon' > 0$ when $\overline{L'_1}$ has a singularity in the (x_1, x_2) -plane. Consider the closure $\overline{L'}$ of L' in \mathbb{R}^3 , which has the structure of a possibly singular minimal lamination. As by construction L' is contained in

$\{x_3 \geq 0\}$, then the already proven part of item 6 of Theorem 1.4 applies to L' and gives that $\overline{L'}$ is contained in a closed slab or halfspace of \mathbb{R}^3 , which must be contained in $\{x_3 \geq 0\}$. Therefore, there exists a collection $\mathcal{P}(L') \subset \overline{L'}$ consisting of one or two planes contained in $\{x_3 \geq 0\}$, such that $\overline{L'} = L' \cup \mathcal{P}(L')$, L' is proper in a component $C(L')$ of $\mathbb{R}^3 - \mathcal{P}(L')$ and $C(L') \cap \widehat{\mathcal{L}} = L'$.

Let $\text{Sing}(\overline{L'})$ be the set of singularities of $\overline{L'}$. Clearly, $\text{Sing}(\overline{L'}) \subset \widehat{\mathcal{S}} \subset \mathcal{S}^B \cup \{\vec{0}\}$. Note that the following properties hold.

- (J1) Through every point $y \in \overline{L'} \cap [\mathcal{S}^B \cup \{\vec{0}\}]$ there passes a plane in $\overline{L'}$: this follows from item 4 of Theorem 1.4 if $y \neq \vec{0}$. In the case that $y = \vec{0}$, then $\vec{0}$ cannot be a singular leaf point of L'_1 by Corollary 2.7; hence L'_1 limits to P and we can take P as the desired plane in $\overline{L'}$ passing through y .
- (J2) Let P' be a horizontal plane contained in $\overline{L'}$ such that $x_3(P') > 0$. Then, $P' \cap [\mathcal{S}^B \cup \text{Sing}(\overline{L'})] = P' \cap \mathcal{S}^B$ is a discrete set by the locally simply connected property of $\widehat{L'}(n)$ in $\{x_3 > 0\}$, and hence it is a finite set (since $P' \cap \mathcal{S}^B$ lies in $C^+(\vec{0}, \tau)$). Actually, $P' \cap \mathcal{S}^B$ consists of at most two points: this follows from a straightforward adaptation of the connecting loop argument in the proof of Lemma 3.3 in [23] using that L'_1 has genus zero.
- (J3) There are no singularities of $\overline{L'}$ in $\{x_3 = 0\} - \{\vec{0}\}$, by property (H).

From properties (J1), (J2) and (J3) we conclude that:

- (J4) $\text{Sing}(\overline{L'}) \cap C(L') = \emptyset$, because there are no planes of $\overline{L'}$ in $C(L')$.
- (J5) Either $P = \{x_3 = 0\}$ is the lower boundary plane of $C(L')$, in which case $\text{Sing}(\overline{L'})$ consists of at most three points, or P lies strictly below the planes in $\mathcal{P}(L')$, in which case $\text{Sing}(\overline{L'})$ consists of at most four points.

Let P' be a plane in $\mathcal{P}(L')$ with positive height, which exists because P_{x_0} exists. The final contradiction that will finish the proof of Claim 3.12 will be a consequence of the following three contradictory properties:

- (K1) $\overline{L'}$ has at most one singularity on P' .
- (K2) $\overline{L'}$ cannot have exactly one singularity on P' .
- (K3) $\overline{L'}$ has at least one singularity on P' .

We now prove (K1) by contradiction: assume that $\overline{L'}$ has two singularities occurring at distinct points q_1, q_2 in a plane $P' \in \mathcal{P}(L')$ at positive height. As the injectivity radius functions of the surfaces $\widehat{L'}(n)$ are uniformly bounded away from zero nearby q_1, q_2 by (9), then the description in (D2) gives that L' has the appearance around the point q_i , $i = 1, 2$, of a disk with the geometry of a spiraling double staircase that limits from above or below to a disk in P' centered at the singularity q_i . The same type of connecting loop argument mentioned in (J2) above together with the fact that the surfaces $\widehat{L'}(n)$ have genus zero imply that the spiraling double staircases at q_1, q_2 are oppositely handed. Furthermore, we can modify slightly the flux-type arguments in the proof of Proposition 4.18 in [23] to find a contradiction in this case; roughly speaking, these arguments

use connecting loops $\Gamma'_n \subset \widehat{L}'(n)$ that converge as $n \rightarrow \infty$ to the twice covered horizontal segment that joins q_1 to q_2 , and lead to a contradiction to the fact that the absolute value of the scalar flux of ∇x_3 along the curves Γ'_n tends to zero as $n \rightarrow \infty$. This proves (K1).

To prove property (K2), suppose that $\overline{L'}$ has exactly one singularity p at a plane $P' \in \mathcal{P}(L')$ at positive height. Suppose for the moment that P' is the lower boundary plane of $C(L')$. Observe that as $x_3(P') > 0$, then there exists a positive constant c' such that for every point $x = (x_1, x_2, x_3) \in \overline{C(L')}$, it holds $|x| \geq c' |x - p|$ (observe that this inequality uses that the distance from x to the origin is greater than some positive number). This inequality and (9) imply that if $x \in \widehat{L}'(n) \cap \{x_3 \geq x_3(P')\}$, then

$$I_{\widehat{L}'(n)}(x) \geq c_1 |x - p|, \quad (11)$$

for $c_1 = c \cdot c' > 0$.

Given $\tau > 0$, consider a sufficiently shallow conical region $C^+(p, \tau)$ defined in (10) so that the singularities of $\overline{L'}$ above P' , if they exist, lie in $C^+(p, 2\tau)$ (recall we have at most two of these singularities, both at the same horizontal plane P'' strictly above P'). It is straightforward to deduce that there exists $a = a(\tau) \in (0, 1)$ such that

$$\text{dist}_{\mathbb{R}^3}(x, \text{Sing}(\overline{L'})) \geq a |x - p| \quad \text{for all } x \in x_3^{-1}([x_3(P'), x_3(P'')]) - C^+(p, \tau), \quad (12)$$

where by convention $x_3(P'') = +\infty$ if P'' does not exist. Clearly, we can assume $c_1 \in (0, a)$ (because inequality (11) stays true if we choose a smaller positive constant c_1).

We claim that the injectivity radius function of L'_1 admits a scale invariant bound of the form

$$\frac{I_{L'_1}(x)}{|x - p|} \geq c_2 > 0 \quad \text{in } L'_1 \cap [\mathbb{R}^3 - C^+(p, \tau)], \quad (13)$$

for some $c_2 > 0$. The proof of (13) is by contradiction: suppose on the contrary that $\frac{I_{L'_1}(x_k)}{|x_k - p|} \rightarrow 0$ as $k \rightarrow \infty$ for some sequence of points $x_k \in L'_1 \cap [\mathbb{R}^3 - C^+(p, \tau)]$. For $k \in \mathbb{N}$ fixed, take a sequence of points $x_{k,n} \in \widehat{L}'(n) \cap \{x_3 > x_3(P')\}$ converging to x_k as $n \rightarrow \infty$ (recall that x_k lies in the regular part L'_1 of L'). Then,

$$I_{\widehat{L}'(n)}(x_{k,n}) \stackrel{(11)}{\geq} c_1 |x_{k,n} - p| \stackrel{(\star)}{>} \frac{c_1}{2} |x_k - p|, \quad (14)$$

where (\star) holds for n sufficiently large. Inequality (14) ensures that the intrinsic metric ball $B_{\widehat{L}'(n)}(x_{k,n}, \frac{c_1}{2} |x_k - p|)$ is a geodesic disk in $\widehat{L}'(n)$, i.e., the exponential map in $\widehat{L}'(n)$ with base point $x_{k,n}$ restricts as a diffeomorphism to the disk of radius $\frac{c_1}{2} |x_k - p|$ centered at the origin in the tangent plane $T_{x_{k,n}} \widehat{L}'(n)$. As $n \rightarrow \infty$, the disks $B_{\widehat{L}'(n)}(x_{k,n}, \frac{c_1}{2} |x_k - p|)$ converge smoothly (with multiplicity one) to the intrinsic metric ball $B_{L'_1}(x_k, \frac{c_1}{2} |x_k - p|)$, because both surfaces $B_{L'_1}(x_k, \frac{c_1}{2} |x_k - p|)$, $B_{\widehat{L}'(n)}(x_{k,n}, \frac{c_1}{2} |x_k - p|)$ are contained for n sufficiently large in the extrinsic ball $\mathbb{B}(x_k, c_1 |x_k - p|)$, and this extrinsic ball does not contain points of $\text{Sing}(\overline{L'})$ by (12) (recall that $0 < c_1 < a$). The continuity of the injectivity radius function under smooth limits (Erllich [12] and Sakai [36]) implies that $I_{L'_1}(x_k) \geq \frac{c_1}{2} |x_k - p|$, which contradicts our hypothesis. This proves our claimed inequality (13) for some $c_2 > 0$.

Observe that as the surfaces $\widehat{L}'(n)$ are locally simply connected away from $\{\vec{0}\}$, then the description (D)-(D2) applies. In particular, L'_1 contains a main component locally around p , whose intersection with $x_3^{-1}(x_3(P'), x_3(P'')) - C^+(p, \tau_1)$ is a pair of ∞ -valued graphs over its projection to the punctured plane $P' - \{p\}$, for some $\tau_1 \in (0, \tau)$. These two ∞ -valued graphs can be connected by curves of uniformly bounded length arbitrarily close to p , by the local description (D2). The scale invariant lower bound (13) for $I_{L'_1}$ is sufficient to apply the arguments in page 45 of Colding and Minicozzi [9]. The existence of such ∞ -valued graphs over the punctured plane $P' - \{p\}$ contradicts Corollary 1.2 in [2], thereby finishing the proof of property (K2) in the particular case that P' is the lower boundary plane of $C(L')$. If P' is the upper boundary plane of $C(L')$ and the lower boundary plane of $C(L')$ is not $P = \{x_3 = 0\}$, then the same reasoning holds after reflecting in P' (because $|x| \geq c'|x - p|$ still holds in this case for some $c' > 0$ and for all $x \in C(L')$). Finally, if P' is the upper boundary plane of $C(L)$ and its lower boundary plane is P , then $|x| \geq c'|x - p|$ holds for each point in $\overline{C(L')} - C^+(p, \tau)$ where δ is chosen so that $\vec{0} \in C^+(p, 2\tau)$ (in other words, with $\vec{0}$ playing the role of the singularities of $\overline{L'}$ above P' of the previous case); therefore, in this last case the arguments above also hold after reflecting in P' . This finishes the proof of property (K2).

To demonstrate property (K3), suppose that $\overline{L'}$ has no singularities in P' . Then, the curvature of L' in a neighborhood of P' is bounded (because the singular set of $\overline{L'}$ is disjoint from P' and the injectivity radius function of L' is bounded away from zero in a neighborhood of P' , as follows from (9) and from the continuity of the injectivity radius function under smooth limits [12, 36], so the one-sided curvature estimates of Colding and Minicozzi apply). In this situation, Lemma 1.4 in [32] together with the proof of the halfspace theorem produce a contradiction, which proves (K3). Now Claim 3.12 follows. \square

Claim 3.13 *Given $\{\lambda_n\}_n \subset (1, \infty)$ with $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, the sequence $\{\lambda_n L_1(\varepsilon)\}_n$ converges to the foliation of the closed upper halfspace of $\mathbb{R}^3 - \{\vec{0}\}$ by horizontal planes, and the singular set of convergence of $\{\lambda_n L_1(\varepsilon)\}_n$ is the positive x_3 -axis.*

Proof. Assume that $\tau, \delta > 0$ are chosen sufficiently small so that the tangent planes to points of L_1 make an angle of less than $\frac{\pi}{4}$ with the horizontal at points of $L_1 \cap [\mathbb{B}(\delta) - C^+(\vec{0}, \tau)]$; this is possible by the intrinsic one-sided curvature estimates in [8] and our assumption (E2-B). Furthermore, these one-sided curvature estimates and the fact that $L_1(\varepsilon)$ is proper in the $\{0 < x_3 \leq \varepsilon\}$ imply that for any $r \in (0, \frac{\delta}{2})$ fixed and for $\beta > 0$ sufficiently small, each point in the set

$$\alpha_r(\beta) := [L_1(\varepsilon) \cap \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = r^2, x_3 \leq \beta r\}] - C^+(\vec{0}, \tau)$$

lies on a component of $\alpha_r(\beta)$ that is either a graph over the circle $C(r) = P \cap \{x_1^2 + x_2^2 = r^2\}$ or an infinite spiraling “graphical” arc, limiting from above to $C(r)$. Recall that we chose points $p_n \in L_1(\varepsilon)$ that converge to $\vec{0}$ and that are blow-up points on the scale of curvature, around which one has a Colding-Minicozzi picture with a pair of multigraphs that extends sideways by Claim 3.12, for n large, all the way to the cylinder $\{x_1^2 + x_2^2 = r^2\}$ for any $r \in (0, \frac{\delta}{2})$ fixed. Therefore, we deduce that for $\delta > 0$ sufficiently small and for any $r \in (0, \frac{\delta}{2})$ fixed, we have that

$$\alpha_r := (L_1(\varepsilon) \cap \{x_1^2 + x_2^2 = r^2\}) - C^+(\vec{0}, \tau)$$

contains a pair of spiraling “graphical” arcs, each one limiting from above to the circle $C(r)$.

We first show that for $r > 0$ sufficiently small, every component of $L_1(\varepsilon) \cap \mathbb{B}(2r)$ is simply connected. Otherwise for every $m \in \mathbb{N}$ sufficiently large, there exists a simple closed curve $\Gamma_m \subset L_1(\varepsilon) \cap \mathbb{B}(1/m)$ that separates $L_1(\varepsilon)$, Γ_m is homologically nontrivial at one of the sides of $L_1(\varepsilon)$ and Γ_m bounds a complete, noncompact, embedded, stable minimal surface $F_m \subset \{0 < x_3 \leq \varepsilon\}$, which is properly embedded in the closure of a component of $\{0 < x_3 \leq \varepsilon\} - L_1(\varepsilon)$, such that $\Gamma_m \subset \partial F_m \subset [\{x_3 = \varepsilon\} \cap L_1(\varepsilon)] \cup \Gamma_m$. By curvature estimates for stable minimal surfaces, for m sufficiently large, F_m intersects $\{x_1^2 + x_2^2 = r^2\}$ in at least one almost-horizontal circle⁷ and these circles converge to $C(r)$ as $m \rightarrow \infty$. However, for m large F_m would intersect the pair of spiraling arcs in α_r , giving a contradiction. This contradiction proves that every component of $L_1(\varepsilon) \cap \mathbb{B}(2r)$ is simply connected (for $r > 0$ chosen sufficiently small).

We now prove Claim 3.13 by contradiction. Suppose that for some sequence $\lambda_n \rightarrow \infty$, a subsequence of $\{\lambda_n L_1(\varepsilon)\}_n$ does not converge to the foliation of $\{x_3 \geq 0\} - \{\vec{0}\}$ by horizontal planes. Observe that the sequence $\lambda_n L_1(\varepsilon)$ has locally positive injectivity radius in $\mathbb{R}^3 - \{\vec{0}\}$, as we are assuming $I_{L_1} \geq c \cdot |\cdot|$ in this case (E2-B). Also observe that $\vec{0}$ lies in the closure of $\lambda_n L_1(\varepsilon)$ for all n . Applying the already proven conclusions of Theorem 1.4 before the list of items 1,...,7 to the closed countable set $W = \{\vec{0}\}$ and to the sequence of minimal surfaces $\{\lambda_n L_1(\varepsilon)\}_n$, we conclude that there exists a (possibly empty) closed subset S' of $\mathbb{R}^3 - \{\vec{0}\}$ and a (regular) minimal lamination $\hat{\mathcal{L}}'$ of $\mathbb{R}^3 - [\{\vec{0}\} \cup S']$ such that after passing to a subsequence (denoted in the same way), $\{\lambda_n L_1(\varepsilon)\}_n$ converges C^α , $\alpha \in (0, 1)$, to $\hat{\mathcal{L}}'$ in $\mathbb{R}^3 - [\{\vec{0}\} \cup S' \cup S(\hat{\mathcal{L}}')]$, where $S(\hat{\mathcal{L}}') \subset \hat{\mathcal{L}}'$ is the singular set of convergence of the $\lambda_n L_1(\varepsilon)$ to $\hat{\mathcal{L}}'$. The arguments in the proof of Claim 3.12 can be adapted to demonstrate that there are no points of $S' \cup S(\hat{\mathcal{L}}')$ in $\{x_3 > 0\}$, and thus, $\hat{\mathcal{L}}'$ consists of $P - \{\vec{0}\}$ together with a single leaf L' , which is a smooth, nonflat surface that is proper in $\{x_3 > 0\}$. Furthermore, the multiplicity of the convergence of $\{\lambda_n L_1(\varepsilon)\}$ to L' is one, as L' is not flat. Since every component of $L_1(\varepsilon) \cap \mathbb{B}(2r)$ is simply connected for $r > 0$ sufficiently small, a standard lifting argument and the convex hull property imply that L' is simply connected.

As L' does not extend through the origin, then Theorem 2.4 produces a sequence of points $q_n \in L'$ such that $q_n \rightarrow \vec{0}$ and $|K_{L'}|(q_n)|q_n|^2 \rightarrow \infty$ as $n \rightarrow \infty$. Without loss of generality, we can assume that the q_n are points of almost-maximal Gaussian curvature, in the sense of Theorem 1.1 in [30]: in particular, the sequence of translated and rescaled surfaces $\mu_n(L' - q_n)$ converges (after passing to a subsequence) to a helicoid, where $\mu_n = 1/\sqrt{|K_{L'}|(q_n)}$. This limit helicoid is vertical, since the scaled surfaces $\frac{1}{|q_n|}L'$ converge to the foliation of $\{x_3 \geq 0\} - \{\vec{0}\}$ by horizontal planes, as follows from Claim 3.12 adapted to this situation. This implies that nearby q_n , L' has a point y_n where the tangent plane $T_{y_n}L'$ is vertical and the intersection $L' \cap (T_{y_n}L') \cap \mathbb{B}(2)$ contains an analytic arc β_n that is arbitrarily close (for n sufficiently large) in the C^1 norm to the straight line segment $(T_{y_n}L') \cap \{(x_1, x_2, x_3) \mid x_3 = x_3(y_n)\} \cap \mathbb{B}(2)$, and so we may assume that β_n has length less than 5. Also, note that the arcs β_n converge after passing to a subsequence to a straight line segment of length 4 passing through the origin, and that for n large enough, $\tilde{\beta}_n := \beta_n \cap \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq 1\}$ joins the two spiraling arcs in $L' \cap \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = 1\}$.

⁷By curvature estimates, F_m is very flat nearby $\{x_1^2 + x_2^2 = r\}$ for m sufficiently large; as Γ_m is arbitrarily close to height zero but F_m is contained in $\{x_3 > 0\}$, then F_m is very horizontal nearby $\{x_1^2 + x_2^2 = r\}$; hence either $F_m \cap \{x_1^2 + x_2^2 = r\}$ contains an almost-horizontal circle, or it contains an almost-horizontal long spiraling arc. This last possibility can be ruled out as F_m has been constructed by a standard procedure as a limit of area-minimizing surfaces, which cannot have this multigraph appearance.

After replacing by a subsequence, we will assume that the last sentence holds for all $n \in \mathbb{N}$.

Consider the arc $\Gamma \subset L'$ consisting of $\tilde{\beta}_1$ together with the two infinitely spiraling arcs in $L' \cap \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = 1\}$ with the same end points as $\tilde{\beta}_1$ and which limit to the circle $\{x_1^2 + x_2^2 = 1\} \subset P$. Observe that Γ is a proper arc in L' , and since L' is simply connected, then Γ separates L' into two components. Let D be the closure of the component of $L' - \Gamma$ that, near its boundary, lies in $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq 1\}$. Note that D is topologically a disk with one end (the boundary ∂D is a Jordan arc). For every $n \geq 2$, let D_n be the compact subdisk of D bounded by $\tilde{\beta}_1 \cup \tilde{\beta}_n$. Thus, $\{D_n\}_{n \geq 2}$ is a compact increasing exhaustion of D . Assume for the moment that the following property (P) holds and we will finish the proof of Claim 3.13 (we will prove property (P) later).

(P) The function $x \in D \mapsto I_{L'}(x)/|x|$ is bounded.

We next prove that the diameter of D is finite. Let $d_D(\cdot, \cdot)$ denote the Riemannian distance function in D . Arguing by contradiction, we can find sequences of points $a_n, b_n \in D$, such that $d_D(a_n, b_n)$ becomes arbitrarily large. Since $L' - C^+(\vec{0}, \tau)$ consists of multivalued graphs with bounded gradient and the arcs $\tilde{\beta}_n$ have uniformly bounded lengths, then we can assume after replacement that a_n, b_n both lie in $C^+(\vec{0}, \tau)$. Since $\{D_n\}_{n \geq 2}$ is a compact exhaustion of D and the diameter of each D_n is finite, then we can assume, without loss of generality that after choosing a subsequence, $b_n \in D - D_n$ and $d_D(a_n, b_n) > n$. Let σ_n be a smooth arc in D with length less than $d_D(a_n, b_n) + 1$. We claim that the points a_n can also be chosen to diverge in D . Otherwise we may assume that after replacing by a subsequence, for all $n \in \mathbb{N}$, $a_n \in D_{j_0}$ for some $j_0 \in \mathbb{N}$, $j_0 \geq 2$. Thus for each integer $k \in [j_0, n - 1] \in \mathbb{N}$, there is a point $a(n, k) \in \sigma_n \cap (D_{k+1} - D_k)$, and then,

$$n < d_D(a_n, b_n) \leq d_D(a_n, a(n, k)) + d_D(a(n, k), b_n) \leq \text{diameter}(D_{k+1}) + d_D(a(n, k), b_n).$$

As $n - \text{diameter}(D_{k+1})$ can be made arbitrarily large for some choices of k, n with $n \rightarrow \infty$ and $k \in [j_0, n - 1]$, then we conclude that given $k \in \mathbb{N}$, $k \geq 2$, there is an $n(k) \in \mathbb{N}$ such that $d_D(a(n(k), k), b_{n(k)}) > k$. Clearly this shows that we may assume that the sequence a_n can also be chosen to diverge in D . Hence, we now have that the sequences $\{a_n\}_n$ and $\{b_n\}_n$ are both diverging in D and as they lie in $C^+(\vec{0}, \delta)$, then in \mathbb{R}^3 both sequences are converging to $\vec{0}$ as $n \rightarrow \infty$. In particular, property (P) gives that for n large, both $I_{L'}(a_n), I_{L'}(b_n)$ can be taken arbitrarily small. Since the extrinsic distance from a_n to the boundary ∂D is greater than some positive number independent of n , then D contains geodesic arcs parameterized by arc length

$$\gamma_{a_n}: [0, L(\gamma_{a_n})) \rightarrow D, \quad \gamma_{b_n}: [0, L(\gamma_{b_n})) \rightarrow D,$$

starting respectively at a_n, b_n , with finite lengths $L(\gamma_{a_n}) = I_{L'}(a_n), L(\gamma_{b_n}) = I_{L'}(b_n)$ that can be both taken arbitrarily small for n large, and with limiting end points the origin. Fix n large and let $m(n)$ be an integer such that $a_n, b_n \in D(m(n))$. As $D(m(n))$ is a compact minimal disk, then the Gauss-Bonnet formula ensures that both $\gamma_{a_n}, \gamma_{b_n}$ can be assumed to exit $D(m(n))$, which implies that these geodesic arcs must intersect $\tilde{\beta}_{m(n)}$. As the length of $\tilde{\beta}_{m(n)}$ is not larger than 5, then there exists a piecewise smooth path $\Delta_n \subset D$ joining a_n, b_n with length not larger than 6 for n large (Δ_n is a union of arcs in $\gamma_{a_n}, \tilde{\beta}_{m(n)}, \gamma_{b_n}$). This is a contradiction, because the intrinsic

distance in D from a_n to b_n was supposed to be arbitrarily large. Therefore, the diameter of D is finite.

The finiteness of the diameter of D insures that we can join the two multivalued graphs in $L' \cap C^+(\vec{0}, \delta)$ by curves of uniformly bounded lengths, which contradicts Corollary 1.2 in [2] (specifically see the paragraph just after Corollary 1.2 in [2]). This contradiction finishes the proof of Claim 3.13, modulo proving property (P), which we demonstrate next.

First note that L' is not complete (otherwise it would be proper by [8] hence it could not be contained in a halfspace by the halfspace theorem). Therefore, the injectivity radius function $I_{L'}$ is finite valued and continuous on L' . Arguing by contradiction, suppose that $I_{L'}(z_n)/|z_n|$ tends to infinity at a sequence of points $z_n \in D$. By continuity of $I_{L'}/|\cdot|$, the points z_n must leave every compact subset of D . This property and the properness of L' in $\{x_3 > 0\}$ imply that after passing to a subsequence, the z_n converge in \mathbb{R}^3 to a point $z_\infty \in P$, $|z_\infty| \leq 1$.

Suppose that $z_\infty \neq \vec{0}$. Therefore, $I_{L'}(z_n) \rightarrow \infty$ as $n \rightarrow \infty$. By Corollary 0.8 in [8], we can find disks in L' centered at z_n of intrinsic radius arbitrarily large, such that the second fundamental form of such disks is arbitrarily small (for n sufficiently large). This is impossible, as such disks would intersect the spiraling curves in $[L' \cap \{x_1^2 + x_2^2 = 1\}] - C^+(\vec{0}, \tau)$. Therefore, $z_\infty = \vec{0}$. Consider the sequence of rescalings $L''_n := \frac{1}{|z_n|} L'$. Since

$$\frac{I_{L''_n}\left(\frac{1}{|z_n|}x\right)}{\frac{1}{|z_n|}|x|} = \frac{I_{L'}(x)}{|x|},$$

then $I_{L''_n}\left(\frac{1}{|z_n|}z_n\right)$ tends to infinity as $n \rightarrow \infty$. As in the previous case, there exist disks in L''_n centered at $\frac{1}{|z_n|}z_n$ of intrinsic radius arbitrarily large, such that the second fundamental form of such disks is arbitrarily small (for n sufficiently large). This is again impossible, because such disks would intersect the multivalued graphs in $L''_n - C^+(\vec{0}, \tau)$ for n large. Now Claim 3.13 is proved. \square

We next apply Claim 3.13 to find a contradiction in Case (E2-B), that in turn will imply that the genus of $L_1(\varepsilon)$ is infinite (finishing the proof of Assertion 3.11). From Claim 3.13 we deduce that for any $r > 0$ sufficiently small, $L_1(\varepsilon) \cap \mathbb{B}(r)$ has the appearance of a spiraling double staircase limiting from above to the closed horizontal disk $\mathbb{B}(r) \cap P$ minus its center.

Since $L_1(\varepsilon)$ is proper in $\{0 < x_3 \leq \varepsilon\}$, we conclude from the last paragraph that given k isolated points q_1, q_2, \dots, q_k in the singular set $\mathcal{S}' \subset \{x_3 = 0\}$ of the lamination $\overline{L_1(\varepsilon)}$, there exist pairwise disjoint disks $D(q_1, \delta_1), \dots, D(q_k, \delta_k) \subset \{x_3 = 0\}$ such that each of the vertical cylinders $\partial D(q_j, \delta_j) \times (0, \varepsilon]$ intersects $L_1(\varepsilon)$ in two spiraling curves that limit to the horizontal circle $\partial D(q_j, \delta_j) \times \{0\}$. As $L_1(\varepsilon)$ has finite genus, then the proof of Lemma 3.3 in [23] implies that there exist at most two of these isolated singular points of \mathcal{S}' , which in turn implies that \mathcal{S}' contains at most two points (since \mathcal{S}' is closed and countable, the subset of its isolated points is dense in \mathcal{S}' by Baire's Theorem), and that if \mathcal{S}' consists of exactly two points, then we find a contradiction with the flux-type arguments along connecting loops as in the proof of Proposition 4.18 in [23] (also see the proof of property (K1) above). Hence, \mathcal{S}' consists of a unique point, and in this case we find a contradiction as in the last paragraph of the proof of property (K2). This contradiction proves that the genus of $L_1(\varepsilon)$ is infinite provided that Case (E2-B) holds, thereby finishing the proof of Assertion 3.11, which in turn demonstrates Proposition 3.5. \square

In the remainder of this section, we will assume that the surfaces M_n that appear in the statement of Theorem 1.4 have uniformly bounded genus and $\mathcal{S} \cup S(\mathcal{L}) \neq \emptyset$; our goal will be to prove the following statement, which will finish the proof of Theorem 1.4.

Proposition 3.14 *Item 7 of Theorem 1.4 holds.*

Proof. Suppose that the surfaces M_n have uniformly bounded genus and $\mathcal{S} \cup S(\mathcal{L}) \neq \emptyset$. As before, we will organize the proof in assertions.

Assertion 3.15 *Through every point $p \in \mathcal{S} \cup S(\mathcal{L})$, there passes a plane of \mathcal{P} (in particular, $\mathcal{P} \neq \emptyset$).*

Proof. Fix a point $p \in \mathcal{S} \cup S(\mathcal{L}) \stackrel{(3)}{=} \mathcal{S}^A \cup (W \cap \overline{\mathcal{L}})^{\text{sing}} \cup S(\mathcal{L})$. By item 4 of Theorem 1.4, the assertion holds if $p \in \mathcal{S}^A \cup S(\mathcal{L})$; hence in the sequel we will assume that $p \in \mathcal{S} \cap W$. We will discuss two possibilities for p , depending on whether or not p is isolated in $\mathcal{S} \cap W$.

- (L1) Assume p is an isolated point of $\mathcal{S} \cap W$. Arguing by contradiction, suppose no plane of \mathcal{P} passes through p . By item 5 of Theorem 1.4, neither of the conditions 5.1, 5.2 hold. Since 5.1 does not occur and p is isolated in the closed set $\mathcal{S} \cap W$, we can find $\varepsilon > 0$ such that $\mathcal{L} \cap \overline{\mathbb{B}}(p, \varepsilon)$ consists of a finite number of noncompact, connected, smooth, properly embedded minimal surfaces $\{\Sigma_1, \dots, \Sigma_m\}$ in $\overline{\mathbb{B}}(p, \varepsilon) - \{p\}$ (otherwise there would be a limit leaf of $\mathcal{L} \cap (\overline{\mathbb{B}}(p, \varepsilon) - \{p\})$, contradicting 5.1), together with finitely many compact, connected, smooth minimal surfaces with boundary on $\partial\overline{\mathbb{B}}(p, \varepsilon)$. By Corollary 2.7 we have that $m = 1$ and the surface Σ_1 has just one end. Since the surfaces M_n have uniformly bounded genus and converge with multiplicity one to Σ_1 (because 5.2 does not occur at p), then Σ_1 has finite genus not greater than the uniform bound on the genus of the M_n . By the last statement in Corollary 2.7, then Σ_1 extends smoothly across p , contradicting that $p \in \mathcal{S}$.
- (L2) Assume that $p \in \mathcal{S} \cap W$ is not an isolated point of $\mathcal{S} \cap W$. Since $\mathcal{S} \cap W$ is a countable closed set of \mathbb{R}^3 , then p must be a limit of isolated points $p_k \in \mathcal{S} \cap W$. By (L1), there pass planes in \mathcal{P} through the points p_k , $k \in \mathbb{N}$. Our assertion holds in this case by taking limits of these planes. This finishes the proof of Assertion 3.15. \square

Assertion 3.16 $\overline{\mathcal{L}} = \mathcal{P}$ (hence item 7.1 of Theorem 1.4 holds).

Proof. Arguing by contradiction, choose a leaf L of $\overline{\mathcal{L}}$ in $\overline{\mathcal{L}} - \mathcal{P}$. By definition of a leaf of a singular lamination, we can decompose $L = L_1 \cup \mathcal{S}_{L_1}$ where L_1 is a leaf of the related regular lamination $\mathcal{L}_1 = \overline{\mathcal{L}} - \mathcal{S}$ of $\mathbb{R}^3 - \mathcal{S}$ defined in (4), and \mathcal{S}_{L_1} is the set of singular leaf points of L_1 . Since $\mathcal{S} \cup S(\mathcal{L}) \neq \emptyset$, then Assertion 3.15 implies that $\mathcal{P} \neq \emptyset$. By item 6 of Theorem 1.4, L does not intersect $\mathcal{S}^A \cup S(\mathcal{L})$, there exists a subcollection $\mathcal{P}(L)$ of one of two planes in \mathcal{P} such that $\overline{L} = L \cup \mathcal{P}(L)$, L is proper in the open slab or halfspace component $C(L)$ of $\mathbb{R}^3 - \mathcal{P}(L)$ that contains L , $C(L) \cap \overline{\mathcal{L}} = L$, and L_1 has infinite genus. But since the convergence of portions of the M_n to L_1 has multiplicity one (otherwise L_1 and L would be flat), then we conclude L_1 has finite genus. This contradiction finishes the proof of the assertion. \square

We now prove item 7.2 of Theorem 1.4. By Assertion 3.16 we have $\overline{\mathcal{L}} = \mathcal{P}$, which implies that $\mathcal{S} = \emptyset$. Since by hypothesis $\mathcal{S} \cup S(\mathcal{L}) \neq \emptyset$, it follows that $S(\mathcal{L}) \neq \emptyset$.

Assertion 3.17 *Let $P \in \mathcal{P}$ be a plane such that $P \cap S(\mathcal{L}) \neq \emptyset$ (note that P exists by item 4 of Theorem 1.4). Then, $P \cap S(\mathcal{L})$ contains at most two points, and if $P \cap S(\mathcal{L}) = \{p_1, p_2\}$, then the two multivalued graphs occurring inside the surfaces M_n near p_1, p_2 are oppositely handed.*

Proof. Description (D1) implies that $P \cap S(\mathcal{L})$ is discrete in $P - W$. Reasoning by contradiction, suppose that $P \cap S(\mathcal{L})$ contains three isolated points p_1, p_2, p_3 . Let $\Gamma \subset P - W$ be a smooth, embedded compact arc joining p_1 to p_2 and disjoint from $S(\mathcal{L}) - \{p_1, p_2\}$. Note that the corresponding two multivalued graphs forming in the surfaces M_n around the points p_1, p_2 are oppositely handed (otherwise, for n large in a fixed size small neighborhood of Γ in $\mathbb{R}^3 - W$, the surfaces M_n would have unbounded genus, see for example the proof of Lemma 3.3 in [23] and also see the proof of property (K1) above). Using an analogous local picture of the M_n near p_3 , one sees that the handedness of the multivalued graph in M_n near p_3 must be opposite to the handedness of the multivalued graph in M_n near p_1 and near p_2 , which is impossible since they have opposite handedness. Hence the assertion follows. \square

Assertion 3.18 *Item 7.2 of Theorem 1.4 holds.*

Proof. Consider a point $p \in S(\mathcal{L})$. By the local description (D1), it follows that locally around p , the set $S(\mathcal{L})$ is a $C^{1,1}$ arc Γ_p that is orthogonal to a local foliation of disks contained in planes of \mathcal{P} . Thus Γ_p is an open straight line segment orthogonal to the planes in \mathcal{P} . Consider the collection $\mathcal{P}(\Gamma_p)$ of planes in \mathcal{P} that intersect Γ_p . If $\mathcal{P}(\Gamma_p) \cap S(\mathcal{L}) = \Gamma_p$, then item 7.2 of Theorem 1.4 follows. Otherwise, there is a point $q \in [\mathcal{P}(\Gamma_p) \cap S(\mathcal{L})] - \Gamma_p$, and the previous arguments show that there exists a related open line segment $\Gamma_q \subset S(\mathcal{L})$ passing through q . Γ_q is clearly parallel to (and disjoint from) Γ_p . Note that we do not require any maximality on Γ_p or Γ_q as line segments contained in $S(\mathcal{L})$ passing respectively through p or q . Choose a plane $P \in \mathcal{P}$ so that it intersects both Γ_p and Γ_q and P is disjoint from W , which is possible since W is countable. By Assertion 3.17, $P \cap S(\mathcal{L})$ contains exactly two points, one in each segment Γ_p, Γ_q . Thus, there is a related limiting minimal parking garage structure \mathcal{F} in some ε -neighborhood of P (see the first paragraph just after the statement of Theorem 1.4 for an explanation of this limiting parking garage structure). Also, Assertion 3.17 gives that the two multivalued graphs forming in M_n near these two points are oppositely handed. Assertion 3.18 now follows. \square

Assertion 3.19 *If the surfaces M_n are compact with boundary, then item 7.3 of Theorem 1.4 holds (this completes the proof of Proposition 3.14 and of Theorem 1.4).*

Proof. Recall that by Assertion 3.16, $\overline{\mathcal{L}} = \mathcal{P}$ (and thus, $\mathcal{L} = \overline{\mathcal{L}}$). After possibly a rotation, assume that the planes in \mathcal{P} are horizontal. The proof of Assertion 3.18 insures that $S(\mathcal{L})$ consists of a nonempty set of open vertical segments (possibly halflines or lines). By Assertion 3.17, every horizontal plane in \mathcal{P} intersects $S(\mathcal{L})$ in at most two points. We now consider two cases, depending upon whether or not some plane $P \in \mathcal{P}$ intersects $S(\mathcal{L})$ in exactly two points.

(M1) Suppose that some plane $P \in \mathcal{P}$ intersects $S(\mathcal{L})$ in exactly two points.

By the proof of item 7.2 of Theorem 1.4 and after replacing P by another plane, we may assume that there exists an open slab Y containing P , which is foliated by planes in \mathcal{P} , and

such that $Y \cap S(\mathcal{L}) = Y \cap \overline{S(\mathcal{L})}$ consists of two connected, vertical line segments with boundary end points in the boundary ∂Y of Y . We next exchange Y by the largest such open slab; more precisely, let $\Delta(Y)$ be the collection of all open slabs Y' in \mathbb{R}^3 such that $Y \subset Y'$, Y' intersects $\overline{S(\mathcal{L})}$ in two open vertical line segments, and the ends points of each line segment in $Y' \cap \overline{S(\mathcal{L})}$ are contained in $\partial Y'$. Define

$$X = \bigcup_{Y' \in \Delta(Y)} Y'. \quad (15)$$

Note that X is either an open slab in $\Delta(Y)$, an open horizontal halfspace or \mathbb{R}^3 . Furthermore, X intersects $\overline{S(\mathcal{L})}$ in exactly two connected components, which are either vertical segments, rays or lines with boundary end points in ∂X , depending on whether or not X is a slab, a halfspace or \mathbb{R}^3 . Our goal to prove Assertion 3.19 in this case (M1) is to show that $X = \mathbb{R}^3$.

Suppose $X \neq \mathbb{R}^3$ and we will find a contradiction. Since $X \neq \mathbb{R}^3$, we may assume without loss of generality the following properties:

- (M1.1) ∂X contains $P_0 = \{x_3 = 0\}$ as one of its boundary planes and X lies below P_0 .
- (M1.2) X intersects $\overline{S(\mathcal{L})}$ in exactly two connected vertical line segments or rays that have end points $p_1, p_2 \in P_0$. One of these two points, say p_1 , does not lie in the interior of a line segment in $\overline{S(\mathcal{L})}$ (this fact follows from the maximality of X). Note that this implies that either p_1 is isolated as an end point of a maximal segment in $\overline{S(\mathcal{L})}$ (see Figure 6-Up), or there exists a sequence of maximal segments in $\overline{S(\mathcal{L})}$ with end points converging to p_1 (Figure 6-Down).

For $i = 1, 2$ and $\varepsilon > 0$, let $\overline{D}(p_i, \varepsilon) = \overline{\mathbb{B}}(p_i, \varepsilon) \cap P_0$ be the closed disk in P_0 centered at p_i of radius ε . For $k \in \mathbb{N}$, let $C(i, \varepsilon, k) = \overline{D}(p_i, \varepsilon) \times [-k, k]$ denote the related compact solid cylinder in \mathbb{R}^3 , $i = 1, 2$. We also denote by $\delta C(i, \varepsilon, k) = \partial \overline{D}(p_i, \varepsilon) \times [-k, k]$ the side of this cylinder. For a generic fixed small value of $\varepsilon > 0$, we have

$$(N.1) \quad \overline{D}(p_1, \varepsilon) \cap \overline{D}(p_2, \varepsilon) = \emptyset, \text{ and}$$

$$(N.2) \quad \partial \overline{D}(p_1, \varepsilon) \cup \partial \overline{D}(p_2, \varepsilon) \text{ is disjoint for all } n \in \mathbb{N} \text{ from the compact countable set}$$

$$A(\varepsilon, k) := \Pi \left([W \cup \overline{S(\mathcal{L})}] \cap \bigcup_{i=1}^2 C(i, \varepsilon, k) \right) \subset P_0,$$

where Π is the orthogonal projection to P_0 ; hence, for such a value of ε , $\partial \overline{D}(p_1, \varepsilon) \cup \partial \overline{D}(p_2, \varepsilon)$ is at least a positive distance from the compact set $A(\varepsilon, k)$.

Since for each $k \in \mathbb{N}$, $\delta C(1, \varepsilon, k) \cup \delta C(2, \varepsilon, k)$ is also at a positive distance from the closed set $W \cup \overline{S(\mathcal{L})}$, then the compact surfaces M_n converge C^1 as $n \rightarrow \infty$ to a subset of the collection of horizontal planes \mathcal{P} near the compact set $\delta C(1, \varepsilon, k) \cup \delta C(2, \varepsilon, k)$. Also recall that the M_n converge below P_0 to a limiting minimal parking garage structure and so, for $i = 1, 2$ fixed and for n large enough depending on k , $M_n \cap \delta C(i, \varepsilon, k)$ contains a pair of pairwise disjoint, long, almost-horizontal, embedded spiral arcs $\alpha_1^k(i, n)$, $\alpha_2^k(i, n)$, each of which joins a point in the bottom boundary circle of $\delta C(i, \varepsilon, k)$ to another point

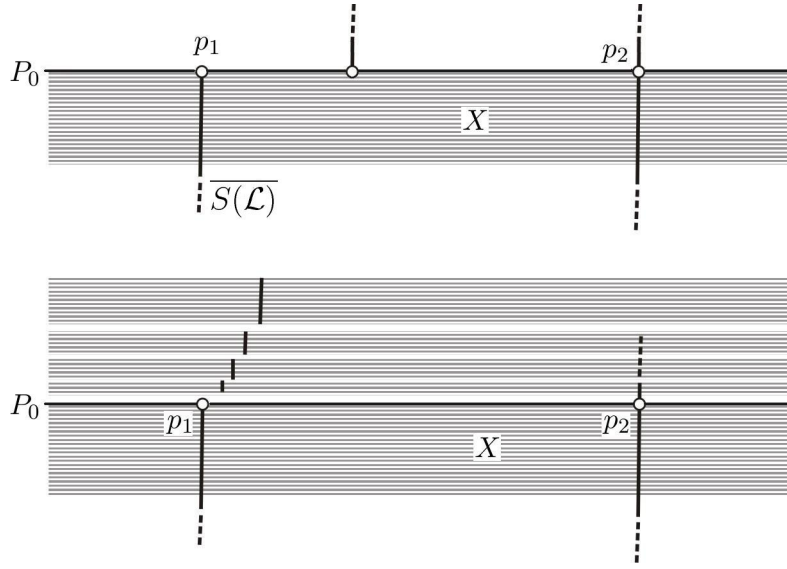


Figure 6: Up: p_1 is isolated as an end point of maximal segments in $\overline{S(\mathcal{L})}$. Down: A sequence of maximal segments in $\overline{S(\mathcal{L})}$ with end points converging to p_1 .

in its top boundary circle, and both $\alpha_1^k(i, n)$, $\alpha_2^k(i, n)$ rotate together around $\delta C(i, \varepsilon, k)$. Furthermore, the Gauss map of M_n along $\alpha_1^k(i, n)$ is arbitrarily close to the north pole of the sphere (for n large) and to the south pole along $\alpha_2^k(i, n)$. Since these spiral arcs intersect every horizontal plane of height between $-k$ and k , then we deduce that the slab $\{-k \leq x_3 \leq k\}$ is foliated by planes in \mathcal{P} . By letting k vary, we deduce that \mathcal{P} is a foliation of \mathbb{R}^3 by horizontal planes.

Fix an integer $k \in \mathbb{N}$. We claim that every horizontal plane $P_t = P_0 + t(0, 0, 1)$, $|t| < k$, intersects $W \cup \overline{S(\mathcal{L})}$ in at least one point of $C(1, \varepsilon, k)$ and in at least one point of $C(2, \varepsilon, k)$. If this intersection property does not hold for some P_t with say $C(1, \varepsilon, k)$, then since $W \cup \overline{S(\mathcal{L})}$ is closed, the compact disk $D_t = P_t \cap C(1, \varepsilon, k)$ is at a positive distance from $W \cup \overline{S(\mathcal{L})}$. Observe that there exists a sequence of minimal disks $E(n, t) \subset C(1, \varepsilon, k)$ with $\partial E(n, t) \subset \delta C(1, \varepsilon, k)$, that are vertical graphs over D_t , converge C^2 to D_t as $n \rightarrow \infty$ and such that for n large, $\partial E(n, t)$ intersects each of the arcs $\alpha_1^k(i, n)$, $\alpha_2^k(i, n)$ transversely in exactly one point (see the perturbation foliation argument in page 737 of [32] for the construction of the boundary curves $\partial E(n, t)$, so that $E(n, t)$ is found by solving the Plateau problem for these curves). After possibly replacing the graphs $E(n, t)$ by sufficiently small vertical translations $E(n, t) + (0, 0, t_n)$ with $t_n \rightarrow 0$, we may assume that $E(n, t)$ intersects M_n transversely in a compact 1-manifold whose boundary is contained in $\partial M_n \cup [M_n \cap \partial E(n, t)]$. Since points in the boundary curves of the surfaces M_n diverge in \mathbb{R}^3 or converge to points in W as $n \rightarrow \infty$ and since for n sufficiently large, $W \cup \overline{S(\mathcal{L})}$ is a positive distance from $E(n, t)$, then also, for n sufficiently large, $M_n \cap E(n, t)$ contains exactly one component with boundary and this component is a compact arc c_n that joins points of the two long spirals $\alpha_1^k(i, n)$, $\alpha_2^k(i, n)$. The property of the Gauss map of M_n along

$\alpha_1^k(i, n) \cup \alpha_2^k(i, n)$ explained in the previous paragraph implies that the normal vector to M_n at some point $q_n \in c_n$ is horizontal. After extracting a subsequence, the points $q_n \in E(n, t)$ converge as $n \rightarrow \infty$ to a point in $[W \cup \overline{S(\mathcal{L})}] \cap P_t \cap C(1, \varepsilon, k)$, which is a contradiction. Hence, our claim holds.

Since W is countable, our claim proved in the previous paragraph implies that every plane P_t , with $|t| < k$, intersects $\overline{S(\mathcal{L})}$ in at least one point of $C(1, \varepsilon, k)$ and in at least one point of $C(2, \varepsilon, k)$. Since, for any $k \in \mathbb{N}$, the generic⁸ radii values ε of $C(1, \varepsilon, k), C(2, \varepsilon, k)$ can be chosen arbitrarily small, it follows that $\overline{S(\mathcal{L})}$ contains every point in the two infinite vertical lines passing through the points p_1, p_2 . By Assertion 3.17, $\overline{S(\mathcal{L})}$ is the union of the two infinite vertical lines passing through the points p_1, p_2 , which proves item 7.3 of Theorem 1.4 holds in case (M1) holds (with two vertical lines for $S(\mathcal{L})$).

(M2) Suppose that every plane in \mathcal{P} intersects $S(\mathcal{L})$ in at most one point.

By item 7.2 of Theorem 1.4, either $\overline{S(\mathcal{L})}$ consists of a single vertical line (and we are done), or there exists a maximal segment in $\overline{S(\mathcal{L})}$ with some end point p . The arguments in the case of (M1) can be easily modified to prove that for every $k \in \mathbb{N}$ and every small generic⁷ radius, the compact cylinder $C(\varepsilon, k) = \overline{D}(p, \varepsilon) \times [-k, k]$ intersects $W \cup \overline{S(\mathcal{L})}$ at any height t with $|t| < k$, and thus, the infinite vertical line l_p passing through p is contained in $\overline{S(\mathcal{L})}$. Since the cases (M1) and (M2) are mutually exclusive, then $\overline{S(\mathcal{L})} = l_p$ and \mathcal{P} is the foliation of \mathbb{R}^3 by horizontal planes.

This finishes the proofs of Assertion 3.19, of Proposition 3.14 and of Theorem 1.4. \square

4 The structure theorem for singular minimal laminations of \mathbb{R}^3 with countable singular set.

We next state the following general structure theorem for possibly singular minimal laminations of \mathbb{R}^3 whose singular set is countable. Theorem 4.1 below is useful in applications because of the following situation. Suppose that L is a nonplanar leaf of a minimal lamination \mathcal{L} of $\mathbb{R}^3 - \mathcal{S}$, with $\mathcal{S} \subset \mathbb{R}^3$ being closed. In this case, its closure \overline{L} has the structure of a possibly singular minimal lamination of \mathbb{R}^3 , which under certain additional hypotheses, can be shown to have a countable singular set. If L has finite genus, then item 6 of the next theorem demonstrates that \overline{L} is a smooth, properly embedded minimal surface in \mathbb{R}^3 , L is the unique leaf of \mathcal{L} and $\mathcal{S} = \emptyset$.

Theorem 4.1 (Structure Theorem for Singular Minimal Laminations of \mathbb{R}^3)

Suppose that $\overline{\mathcal{L}} = \mathcal{L} \cup \mathcal{S}$ is a possibly singular minimal lamination of \mathbb{R}^3 with a countable set \mathcal{S} of singularities. Then:

1. The set \mathcal{P} of leaves in $\overline{\mathcal{L}}$ that are planes forms a closed subset of \mathbb{R}^3 .
2. The set $\text{Lim}(\overline{\mathcal{L}})$ of limit leaves of $\overline{\mathcal{L}}$ forms a closed set in \mathbb{R}^3 and satisfies $\text{Lim}(\overline{\mathcal{L}}) \subset \mathcal{P}$.

⁸In the sense of properties (N.1), (N.2) above.

3. If P is a plane in $\mathcal{P} - \text{Lim}(\overline{\mathcal{L}})$, then there exists $\delta > 0$ such that $P(\delta) \cap \overline{\mathcal{L}} = P$, where $P(\delta)$ is the δ -neighborhood of P . In particular, $\mathcal{S} \cap [\mathcal{P} - \text{Lim}(\overline{\mathcal{L}})] = \emptyset$.
4. Suppose $p \in \mathcal{S} - \bigcup_{P \in \mathcal{P}} P$. Then for almost all $\varepsilon > 0$ sufficiently small, $\mathcal{L}(p, \varepsilon) = \mathcal{L} \cap \overline{\mathbb{B}}(p, \varepsilon)$ has the following description.
 - 4.1. $\mathcal{L}(p, \varepsilon)$ consists of a finite number of leaves, each of which is a properly embedded smooth surface in $\overline{\mathbb{B}}(p, \varepsilon) - \mathcal{S}$ with compact boundary in $\mathbb{S}^2(p, \varepsilon)$.
 - 4.2. All of the leaves of $\mathcal{L}(p, \varepsilon)$ lie on the same leaf of $\overline{\mathcal{L}}$.
 - 4.3. Each point $q \in \mathbb{B}(p, \varepsilon) \cap \mathcal{S}$ represents the end of a unique leaf L_q of $\mathcal{L}(p, \varepsilon)$, in the sense that there is a proper arc $\alpha: [0, 1) \rightarrow L_q$ with $q = \lim_{t \rightarrow 1} \alpha(t)$. Furthermore, this end of L_q has infinite genus ($L_q = L_{q'}$ may occur if q, q' are distinct points in $\mathbb{B}(p, \varepsilon) \cap \mathcal{S}$, for example this occurs if $\mathbb{B}(p, \varepsilon) \cap \mathcal{S}$ is infinite for all small $\varepsilon > 0$). In fact, if p is an isolated point of \mathcal{S} , then ε can be chosen small enough so that $\mathcal{L}(p, \varepsilon)$ is contained in the leaf of \mathcal{L} that contains L_p , and L_p has infinite genus and exactly one end.

In items 5, 6 below, suppose that $L = \overline{\mathcal{L}}(L_1) = L_1 \cup \mathcal{S}_{L_1}$ is a leaf of $\overline{\mathcal{L}}$ that is not contained in \mathcal{P} , where L_1 is the related leaf of \mathcal{L} , and \mathcal{S}_{L_1} is the set of singular leaf points of L_1 .

5. One of the following possibilities holds.

- 5.1. L is proper in \mathbb{R}^3 , and L is the unique leaf of \mathcal{L} .
- 5.2. L is not proper in \mathbb{R}^3 and $\mathcal{P} \neq \emptyset$. In this case, the closure \overline{L} of L in \mathbb{R}^3 has the structure of a possibly singular minimal lamination of \mathbb{R}^3 (with singular set contained in $\overline{L} \cap \mathcal{S}$) and there exists a subcollection $\mathcal{P}(L) \subset \mathcal{P}$ consisting of one or two planes, such that $\overline{L} = L \cup \mathcal{P}(L)$ and L is proper in a component $C(L)$ of $\mathbb{R}^3 - \mathcal{P}(L)$ and $C(L) \cap \overline{\mathcal{L}} = L$. Furthermore (see Figure 7):
 - a. Every open ε -neighborhood $P(\varepsilon)$ of a plane $P \in \mathcal{P}(L)$ intersects L_1 in a connected surface with unbounded Gaussian curvature.
 - b. If some open ε -neighborhood $P(\varepsilon)$ of a plane $P \in \mathcal{P}(L)$ intersects L_1 in a surface with finite genus, then $P(\varepsilon)$ is disjoint from the singular set of \overline{L} .
 - c. L_1 has infinite genus.

In particular, $\overline{\mathcal{L}}$ is the disjoint union of its leaves, regardless of whether case 5.1 or 5.2 occurs.

6. If L_1 has finite genus, then $L = L_1$ is a smooth, properly embedded minimal surface in \mathbb{R}^3 (thus $\mathcal{L} = \overline{\mathcal{L}}$, L is the unique leaf of \mathcal{L} and $\mathcal{S} = \emptyset$).

As in the previous section, we will divide the proof of Theorem 4.1 into several lemmas.

Lemma 4.2 *Items 1, 2 and 3 of Theorem 4.1 hold.*

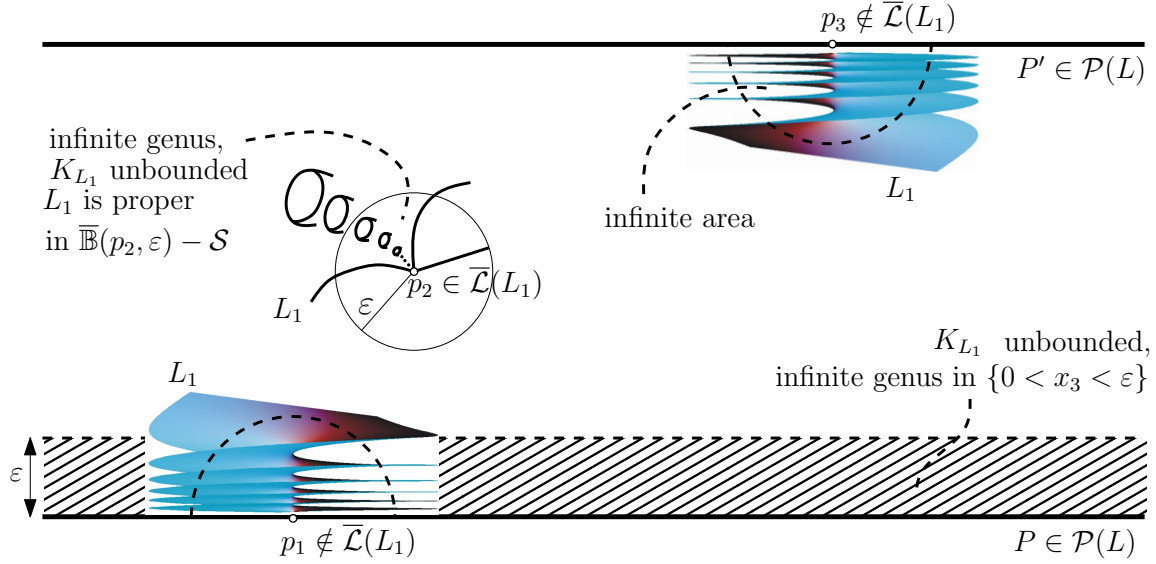


Figure 7: Behavior of any nonproper leaf $L = L_1 \cup S_{L_1}$ of the singular lamination $\bar{\mathcal{L}}$ in item 5.2 of Theorem 4.1: L_1 has infinite genus in any neighborhood of a singularity of type p_2 and in any slab neighborhood of a plane containing a singularity of type p_1 , and no other leaves of $\bar{\mathcal{L}}$ can occur between P, P' .

Proof. The proof of item 1 of Theorem 4.1 is the same as the one of item 1 of Theorem 1.4. As for the proof of item 2 of Theorem 4.1, the second paragraph of the proof of Lemma 3.1 can be applied without changes to show that if $L = L_1 \cup S_{L_1}$ is a limit leaf of $\bar{\mathcal{L}}$, then the two-sided cover of L_1 is stable. By items 1 and 2 of Corollary 2.5, L_1 extends across the countable set \mathcal{S} to a plane. Hence, L is a plane itself. Since the limit of planes in $\text{Lim}(\bar{\mathcal{L}})$ is clearly a limit leaf of \mathcal{L} , it follows that $\text{Lim}(\bar{\mathcal{L}})$ is closed in \mathbb{R}^3 . These observations prove item 2 of Theorem 4.1.

Next we prove item 3 of Theorem 4.1. The argument will be similar to that in the proof of Lemma 3.3. Let P be a plane in $\mathcal{P} - \text{Lim}(\bar{\mathcal{L}})$. By item 2 of Theorem 4.1, we can choose $\delta > 0$ such that the 2δ -neighborhood $P(2\delta)$ of P does not intersect $\cup_{P' \in \mathcal{P} - \{P\}} P'$. Suppose, arguing by contradiction, that the closed slab $\bar{P}(\delta)$ intersects $\bar{\mathcal{L}}$ in a portion of a leaf L of $\bar{\mathcal{L}}$ different from P . Note that as a set, $L \cap \bar{P}(\delta)$ is proper in $\bar{P}(\delta)$ (otherwise we produce a limit leaf of $\bar{\mathcal{L}}$ in $\bar{P}(\delta)$, hence a plane which cannot be P , as $P \notin \text{Lim}(\bar{\mathcal{L}})$). Proposition 2.6 implies that L is disjoint from P . Now, a standard application of the proof of the halfspace theorem [16]) using catenoid barriers still works in this setting to obtain a contradiction to the existence of L , thereby finishing the proof of Lemma 4.2. \square

Lemma 4.3 *Item 4 of Theorem 4.1 holds.*

Proof. Take a point $p \in \mathcal{S} - \cup_{P \in \mathcal{P}} P$. By item 1 of Theorem 4.1, we can choose $\varepsilon' > 0$ small enough so that $\mathbb{B}(p, 3\varepsilon')$ does not intersect \mathcal{P} ; hence $\mathbb{B}(p, 3\varepsilon')$ does not intersect $\text{Lim}(\bar{\mathcal{L}})$ as well, by item 2 of Theorem 4.1. This last property implies that leaves of \mathcal{L} are proper in $\mathbb{B}(p, 2\varepsilon')$.

Consider a number $\varepsilon \in (\varepsilon', 2\varepsilon')$ such that the sphere $\mathbb{S}^2(p, \varepsilon)$ is at a positive distance from \mathcal{S} and is transverse to \mathcal{L} . Therefore, $\mathbb{S}^2(p, \varepsilon)$ intersects \mathcal{L} in a finite number of smooth closed curves. Since every component of $\mathcal{L}(p, \varepsilon) = \mathcal{L} \cap \overline{\mathbb{B}}(p, \varepsilon)$ intersects $\mathbb{S}^2(p, \varepsilon)$ (a leaf L_1 of $\mathcal{L}(p, \varepsilon)$ completely contained in $\overline{\mathbb{B}}(p, \varepsilon)$ would contradict Proposition 2.6 applied to L_1 and to a plane passing through a point in L_1 at maximum distance from p), then we conclude that item 4.1 of Theorem 4.1 holds.

To prove item 4.2, note that as all of the components of $\mathcal{L}(p, \varepsilon)$ are proper as sets in $\overline{\mathbb{B}}(p, \varepsilon) - \mathcal{S}$, then p is a singular leaf point of any leaf of $\overline{\mathcal{L}} \cap \overline{\mathbb{B}}(p, \varepsilon)$ that has p in its closure. By Proposition 2.6, only one of the components of $\mathcal{L}(p, \varepsilon)$, say $C(p, \varepsilon)$, has p in its closure. Hence, we can reduce ε to $\varepsilon_1 > 0$ so that $\mathcal{L}(p, \varepsilon_1) \subset C(p, \varepsilon)$ and item 4.2 is proved.

Regarding item 4.3, its first statement follows from Proposition 2.6. Recall that if e is an end of a noncompact surface Σ and $\alpha: [0, 1) \rightarrow \Sigma$ is a proper arc representing e , then e has infinite genus if every proper subdomain $\Omega \subset \Sigma$ with compact boundary that contains the end of α , has infinite genus. To prove the second statement in item 4.3 we argue by contradiction: take $q \in \overline{\mathbb{B}}(p, \varepsilon) \cap \mathcal{S}$ and let Σ be the component of $\mathcal{L}(p, \varepsilon)$ that contains the point q . Suppose that α is a proper arc representing the end of Σ corresponding to q , such that Σ contains a proper subdomain Ω with finite genus and compact boundary, in such a way that the end of α is contained in Ω . Choose $\delta \in (0, \varepsilon)$ sufficiently small so that $\partial\Omega$ lies outside $\overline{\mathbb{B}}(q, \delta) \subset \overline{\mathbb{B}}(p, \varepsilon)$ and $\partial\overline{\mathbb{B}}(q, \delta)$ is transverse to Σ . Let Ω' be the component of $\Omega \cap \overline{\mathbb{B}}(q, \delta)$ that contains the end of α . Since Ω' is properly embedded in $\overline{\mathbb{B}}(q, \delta) - \mathcal{S}$, then the set of points $\overline{\Omega'} \cap \mathcal{S}$ is a nonempty closed countable subset of $\overline{\mathbb{B}}(q, \delta)$. Baire's Theorem implies that the set of isolated singularities in $\overline{\Omega'} \cap \mathcal{S}$ is dense in $\overline{\Omega'} \cap \mathcal{S}$. But Corollary 2.7 applied around an isolated singularity of $\overline{\Omega'} \cap \mathcal{S}$ in $\overline{\mathbb{B}}(q, \delta) \cap \mathcal{S}$ gives a contradiction since Ω' has finite genus. This contradiction completes the proof of the second statement in item 4.3. Finally, the last statement in item 4.3 is a consequence of the previously proved parts of this item. Now the lemma holds. \square

Proposition 4.4 *Items 5, 6 of Theorem 4.1 hold (and so, the proof of Theorem 4.1 is complete).*

Proof. Suppose that $L = L_1 \cup \mathcal{S}_{L_1}$ is a leaf of \mathcal{L} not contained in \mathcal{P} (with the notation of Theorem 4.1). Following the reasoning in the proof of Proposition 3.5, we will distinguish two cases, depending on whether or not L is proper as a set in \mathbb{R}^3 . If L is proper in \mathbb{R}^3 , then the arguments in case (E1) of the proof of Proposition 3.5 are now valid and prove that item 5.1 of Theorem 4.1 holds.

Now assume that L is not proper in \mathbb{R}^3 , and we will deduce that item 5.2 of Theorem 4.1 holds. As before, we will only comment on how to adapt the arguments in (E2) of the proof of Proposition 3.5 to our current setting. The property that through every limit point of L there passes a plane in \mathcal{P} (that is, Assertion 3.7) follows verbatim, with the only change of \mathcal{L}_1 by \mathcal{L} in the proof of Assertion 3.7. This implies that $\overline{L} = L \cup \mathcal{P}(L)$ with $\mathcal{P}(L) \subset \mathcal{P}$ consisting of one or two planes, and L is proper in the component $C(L)$ of $\mathbb{R}^3 - \mathcal{P}(L)$ that contains L . Assertion 3.8 also holds in our new setting, with the only change in its proof occurring when demonstrating the countability of the set of points of $\overline{\Sigma}$ where the least-area surface Σ is possibly incomplete, which is easier now as this set is clearly contained in the countable set $\mathcal{S} \cap [L \cup L' \cup \mathcal{P}(L)]$. Assertion 3.9 also holds true now, with the only change in its proof that incompleteness of the surface $L_1(\varepsilon) = L_1 \cap \{0 < x_3 \leq \varepsilon\}$ (we assume the same normalization as at the beginning of

the proof of Assertion 3.9) may fail at the set $\mathcal{S} \cap \{0 \leq x_3 \leq \varepsilon\}$, which is countable. Hence, the proof of the main statement of item 5.2 and item 5.2(a) of Theorem 4.1 are proved.

Assertion 3.11 and its proof are valid in our current setting without changes, as all their arguments rely on the limit singular lamination $\overline{\mathcal{L}}$ of \mathbb{R}^3 and not in the sequence of minimal surfaces $\{M_n\}_n$ that appear in the statement of Theorem 1.4. Therefore, items 5.2(b) and 5.2(c) of Theorem 4.1 are also proved.

Finally, item 6 of Theorem 4.1 follows directly from item 5 of the same theorem. \square

5 A convergence result for embedded minimal surfaces of uniformly bounded genus.

Theorem 5.1 *Suppose $\{M_n\}_n$ is a sequence of compact, embedded minimal surfaces of finite genus at most $g \in \mathbb{N} \cup \{0\}$, with $\partial M_n \subset \mathbb{S}^2(n)$ for each n . Suppose that some subsequence of disks $\{D_n \subset M_n\}_n$ converges C^2 to a nonflat minimal disk. Then, a subsequence of the M_n converges smoothly on compact subsets of \mathbb{R}^3 with multiplicity one to a connected, properly embedded, nonflat minimal surface $M_\infty \subset \mathbb{R}^3$ of genus at most g , that is either a surface of finite total curvature, a helicoid with handles or a two-limit-ended minimal surface. Furthermore, M_∞ has bounded Gaussian curvature.*

Proof. Suppose for the moment that there exists $R > 0$ such that

$$\sup_{M_n \cap \mathbb{B}(R)} |K_{M_n}| \rightarrow \infty \text{ as } n \rightarrow \infty.$$

By Theorem 0.6 in [9] (see also Footnote 3 in the statement of that theorem), then after a rotation in \mathbb{R}^3 , there exists a subsequence of these compact minimal surfaces, also denoted by $\{M_n\}_n$, a lamination $\mathcal{L}_1 = \{x_3 = t\}_{t \in \mathcal{I}}$ by parallel planes (where $\mathcal{I} \subset \mathbb{R}$ is a closed set), and a nonempty closed set $S(\mathcal{L}_1)$ in the union of the leaves of \mathcal{L}_1 such that:

(N1) For each $\alpha \in (0, 1)$, $\{M_n - S(\mathcal{L}_1)\}_n$ converges in the C^α -topology to the lamination $\mathcal{L}_1 - S(\mathcal{L}_1)$.

(N2) $\sup_{M_n \cap \mathbb{B}(x, r)} |K_{M_n}| \rightarrow \infty$ as $n \rightarrow \infty$ for all $x \in S(\mathcal{L}_1)$ and $r > 0$.

Our hypothesis that a sequence of disks $D_n \subset M_n$ converges C^2 to a nonflat minimal disk contradicts that \mathcal{L}_1 consists of planar leaves. This contradiction shows that the M_n have uniformly bounded Gaussian curvatures on compact subsets of \mathbb{R}^3 . Therefore, a standard diagonal argument implies that after passing to a subsequence, the M_n converge to a (regular) minimal lamination \mathcal{L} of \mathbb{R}^3 .

By the structure theorem for (regular) minimal laminations of \mathbb{R}^3 (see Theorem 1.6 in [32]), the collection \mathcal{P} of planes in \mathcal{L} forms a possibly empty, closed set of \mathbb{R}^3 , each of the components X of $\mathbb{R}^3 - \mathcal{P}$ contains at most one leaf L_X of \mathcal{L} , and such a leaf L_X is not flat and proper in X . Since every such a L_X is nonflat, its universal cover cannot be stable and the proof of Lemma A.1 in [33] implies that the multiplicity of the convergence of portions of the M_n to L_X is one. In this

setting, standard lifting arguments give that one can lift any handle on L_X to a nearby handle on an approximating surface M_n for n large, such that any fixed finite collection of pairwise disjoint handles in L_X lifts to a collection of disjoint handles on the nearby surface M_n . Since the genus of each M_n is at most g , then L_X has genus at most g . By Corollary 1 in [24], L_X is the only leaf in \mathcal{L} and L_X is properly embedded in \mathbb{R}^3 . This proves the first item in Theorem 5.1 with M_∞ being L_X .

By Theorem 1 in [25], the surface M_∞ is either a surface of finite total curvature, a helicoid with handles or a minimal surface with two limit ends. The same theorem states that M_∞ has bounded curvature, which completes the proof of Theorem 5.1. \square

William H. Meeks, III at profmeeks@gmail.com

Mathematics Department, University of Massachusetts, Amherst, MA 01003

Joaquín Pérez at jperez@ugr.es

Antonio Ros at aros@ugr.es

Department of Geometry and Topology and Institute of Mathematics (IEMath-GR), University of Granada, 18071, Granada, Spain

References

- [1] W. K. Allard. On the first variation of a varifold. *Ann. of Math.*, 95:417–491, 1972. MR0307015, Zbl 0252.49028.
- [2] T. H. Colding and W. P. Minicozzi II. Multivalued minimal graphs and properness of disks. *International Mathematical Research Notices*, 21:1111–1127, 2002. MR1904463, Zbl 1008.58012.
- [3] T. H. Colding and W. P. Minicozzi II. Embedded minimal disks: proper versus nonproper - global versus local. *Transactions of the AMS*, 356(1):283–289, 2003. MR2020033, Zbl 1046.53001.
- [4] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold I; Estimates off the axis for disks. *Ann. of Math.*, 160:27–68, 2004. MR2119717, Zbl 1070.53031.
- [5] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold II; Multi-valued graphs in disks. *Ann. of Math.*, 160:69–92, 2004. MR2119718, Zbl 1070.53032.
- [6] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold III; Planar domains. *Ann. of Math.*, 160:523–572, 2004. MR2123932, Zbl 1076.53068.
- [7] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold IV; Locally simply-connected. *Ann. of Math.*, 160:573–615, 2004. MR2123933, Zbl 1076.53069.

- [8] T. H. Colding and W. P. Minicozzi II. The Calabi-Yau conjectures for embedded surfaces. *Ann. of Math.*, 167:211–243, 2008. MR2373154, Zbl 1142.53012.
- [9] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold V ; Fixed genus. *Ann. of Math.*, 181(1):1–153, 2015. MR3272923, Zbl 06383661.
- [10] P. Collin. Topologie et courbure des surfaces minimales de \mathbb{R}^3 . *Ann. of Math. (2)*, 145–1:1–31, 1997. MR1432035, Zbl 886.53008.
- [11] P. Collin, R. Kusner, W. H. Meeks III, and H. Rosenberg. The geometry, conformal structure and topology of minimal surfaces with infinite topology. *J. Differential Geom.*, 67:377–393, 2004. MR2153082, Zbl 1098.53006.
- [12] P. E. Ehrlich. Continuity properties of the injectivity radius function. *Compositio Math.*, 29:151–178, 1974. MR0417977, Zbl 0289.53034.
- [13] A. Grigor’yan. Analytic and geometric background of recurrence and non-explosion of Brownian motion on Riemannian manifolds. *Bull. of A.M.S.*, 36(2):135–249, 1999. MR1659871, Zbl 0927.58019.
- [14] M. Grüter. Regularity of weak h -surfaces. *J. Reine Angew. Math.*, 329:1–15, 1981. MR0636440, Zbl 0461.53029.
- [15] R. Harvey and B. Lawson. Extending minimal varieties. *Invent. Math.*, 28:209–226, 1975. MR0370319, Zbl 0316.49032.
- [16] D. Hoffman and W. H. Meeks III. The strong halfspace theorem for minimal surfaces. *Invent. Math.*, 101:373–377, 1990. MR1062966, Zbl 722.53054.
- [17] F. J. López and A. Ros. On embedded complete minimal surfaces of genus zero. *J. Differential Geom.*, 33(1):293–300, 1991. MR1085145, Zbl 719.53004.
- [18] W. H. Meeks III. The regularity of the singular set in the Colding and Minicozzi lamination theorem. *Duke Math. J.*, 123(2):329–334, 2004. MR2066941, Zbl 1086.53005.
- [19] W. H. Meeks III. The limit lamination metric for the Colding-Minicozzi minimal lamination. *Illinois J. of Math.*, 49(2):645–658, 2005. MR2164355, Zbl 1087.53058.
- [20] W. H. Meeks III and J. Pérez. The classical theory of minimal surfaces. *Bulletin of the AMS*, 48:325–407, 2011. MR2801776, Zbl 1232.53003.
- [21] W. H. Meeks III, J. Pérez, and A. Ros. Bounds on the topology and index of classical minimal surfaces. Preprint available at <https://arxiv.org/abs/1605.02501>.
- [22] W. H. Meeks III, J. Pérez, and A. Ros. The embedded Calabi-Yau conjectures for finite genus. Work in progress.
- [23] W. H. Meeks III, J. Pérez, and A. Ros. The local picture theorem on the scale of topology. To appear in *J. Differential Geometry*. Preprint at <https://arxiv.org/abs/1505.06761>.

- [24] W. H. Meeks III, J. Pérez, and A. Ros. The geometry of minimal surfaces of finite genus I; curvature estimates and quasiperiodicity. *J. Differential Geom.*, 66:1–45, 2004. MR2128712, Zbl 1068.53012.
- [25] W. H. Meeks III, J. Pérez, and A. Ros. The geometry of minimal surfaces of finite genus II; nonexistence of one limit end examples. *Invent. Math.*, 158:323–341, 2004. MR2096796, Zbl 1070.53003.
- [26] W. H. Meeks III, J. Pérez, and A. Ros. Stable constant mean curvature surfaces. In *Handbook of Geometrical Analysis*, volume 1, pages 301–380. International Press, edited by Lizhen Ji, Peter Li, Richard Schoen and Leon Simon, ISBN: 978-1-57146-130-8, 2008. MR2483369, Zbl 1154.53009.
- [27] W. H. Meeks III, J. Pérez, and A. Ros. Limit leaves of an H lamination are stable. *J. Differential Geom.*, 84(1):179–189, 2010. MR2629513, Zbl 1197.53037.
- [28] W. H. Meeks III, J. Pérez, and A. Ros. Properly embedded minimal planar domains. *Ann. of Math.*, 181(2):473–546, 2015. MR3275845, Zbl 06399442.
- [29] W. H. Meeks III, J. Pérez, and A. Ros. The classification of *CMC* foliations of \mathbb{R}^3 and \mathbb{S}^3 with countably many singularities. *American J. of Math.*, 138(5):1347–1382, 2016. Preprint at <http://arxiv.org/pdf/1401.2813>.
- [30] W. H. Meeks III, J. Pérez, and A. Ros. The Dynamics Theorem for properly embedded minimal surfaces. *Mathematische Annalen*, 365(3):1069–1089, 2016. MR3521082, Zbl 06618524.
- [31] W. H. Meeks III, J. Pérez, and A. Ros. Local removable singularity theorems for minimal laminations. *J. Differential Geometry*, 103(2):319–362, 2016. MR3504952, Zbl 06603546.
- [32] W. H. Meeks III and H. Rosenberg. The uniqueness of the helicoid. *Ann. of Math.*, 161:723–754, 2005. MR2153399, Zbl 1102.53005.
- [33] W. H. Meeks III and H. Rosenberg. The minimal lamination closure theorem. *Duke Math. Journal*, 133(3):467–497, 2006. MR2228460, Zbl 1098.53007.
- [34] W. H. Meeks III, L. Simon, and S. T. Yau. Embedded minimal surfaces, exotic spheres and manifolds with positive Ricci curvature. *Ann. of Math.*, 116:621–659, 1982. MR0678484, Zbl 0521.53007.
- [35] W. H. Meeks III and S. T. Yau. The existence of embedded minimal surfaces and the problem of uniqueness. *Math. Z.*, 179:151–168, 1982. MR0645492, Zbl 0479.49026.
- [36] T. Sakai. On continuity of injectivity radius function. *Math. J. Okayama Univ.*, 25(1):91–97, 1983. MR701970, Zbl 0525.53053.